"The backwards ones?" - Undergraduate students' reactions and approaches to example generation exercises

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As part of a project exploring the design and use of mathematical tasks to promote conceptual understanding of Calculus concepts, first year undergraduate students were assigned homework problems which required them to use various processes including generalising, conjecturing, evaluating statements, analysing reasoning and generating examples. In subsequent interviews with five students, a number of them spontaneously referred to the example generation problems posed as being the "backwards ones" or requiring them to work backwards as well as forwards. In this paper, we will report on the students' reactions to a particular example generation exercise, the strategies they adopted in an effort to solve such problems, and what they feel they learnt in the process.

Keywords: example generation tasks, conceptual understanding, interview data.

Introduction

It has been found that mathematicians use examples in a number of ways: to help them understand a statement or definition, to help them generate an argument, to help decide whether a statement is in fact false (Alcock 2004) and to increase their confidence in a particular result (Weber and Mejia-Ramos 2011). Many people have spoken about the importance of specific examples in aiding understanding of general results. Feynman famously said

> I can't understand anything in general unless I'm carrying along in my mind a specific example and watching it go. (Feynmann, Leighton and Hutchings 1997 (244)

Many authors have advised that mathematics lessons should incorporate activities that invoke the use of mathematical habits of mind in learners in order to allow learners to experience the practices of mathematicians (e.g. Bass 2005). Cuoco, Goldenberg and Mark advocate the inclusion of such habits of mind in order to "give the students the tools they will need in order to use, understand and even make the mathematics that does not yet exist" (1996, 376). Thus, it would seem sensible to encourage students to use examples and to generate their own examples to help them understand definitions and results. Some authors have discussed the contribution examples make in enabling students to develop a better-informed concept image (Meehan 2002). Other researchers have spoken about example generation exercises as transferring the initiative to the learner (Bills et al. 2006). Such exercises provide an opportunity for learners to be assertive and creative, to exercise choice. In being constructive in this way, students can experience freedom and constraint, and can identify and explore the dimensions of variation possible in the choice of examples available, as well as come to an appreciation of aspects of invariance in the midst of change (Mason and Johnston-Wilder 2004; Mason and Watson 2002).

Hazzan and Zazkis (1999) identified a number of common approaches taken by students when faced with example generation exercises: some use trial and error, others attempt to use an algorithm or, when one is not available, attempt to create one for themselves, while others begin with a trivial specific example and try to modify it to obtain more sophisticated examples. Edwards (2011) discusses a number of studies reporting similar findings.

Task design project

The example generation exercises which are the subject of this paper were assigned to first year undergraduate Calculus students at an Irish university as part of a project focussing on the design and implementation of mathematical tasks that aim to promote conceptual understanding and to encourage the use of mathematical ‘habits of mind’ amongst students. There has been very little research on mathematics education at university level in Ireland, but research on mathematics education at the senior cycle of post-primary school has identified a predominantly procedural approach to mathematics teaching and learning. The chief examiner at the State Examinations Commission has found that students’ performance on the state examination at the end of post-primary school showed they had inadequate understanding of concepts (SEC 2005). Others have found the state examinations are predictable and reward the learning of rules and their application in familiar contexts (Elwood and Carlisle 2003). Because of the backwash effect of assessment on teaching and learning, “shaping both what is taught and how it is taught” (Conway and Sloane 2005, 28), it has been found that Irish classrooms tend to be focussed on the use of algorithmic procedures, with very little emphasis on conceptual understanding, and that students appear unable to apply techniques learnt in unfamiliar contexts (Lyons et al. 2003; Hourigan and O’Donoghue 2007).

Given this context, the authors identified the following types of tasks as being appropriate for Irish first year undergraduate Calculus students and designed a number of such tasks: tasks requiring students to generate examples, evaluate statements, analyse reasoning, conjecture, generalise, visualise, and/or use definitions. The tasks designed are being evaluated through the collection of data from the students using a variety of means, one of which is the interviewing of a small sample of students.

Methodology

In the first semester of the academic year 2011/12, the second author piloted some of the tasks designed in this project both as homework assignments (for students to work on independently) and as tutorial problems (for students to work on in small groups). The homework assignments were graded and contributed to the continuous assessment marks for this NUI Maynooth Differential Calculus module. At the end of the semester, all 130 registered students were emailed and invited to participate in interviews with the third author; eight students responded and because of timetable constraints five students were interviewed. The abilities of these students (as measured by a diagnostic test at the beginning of the semester and by a midterm test) were spread across the spectrum. Three of the students were interviewed individually (Students C, P and T), and two were interviewed together (Students A and E). The interviews were semi-structured and of approximately 15 minutes duration; they were audio-recorded and were fully transcribed by the third author. All three authors coded the transcripts separately and then met to discuss the codes.

Results

In the interviews, when students were asked to comment on the different tasks they had been assigned, a number of them chose to talk about one particular example generation exercise. This task is shown below.

Question A: Find examples of the following:

(a) Polynomials P(x) and Q(x) such that P(4) = 0 = Q(4) and $\lim_{x \to 4} \frac{P(x)}{Q(x)} = 0$

(b) Polynomials P(x) and Q(x) such that P(4) = 0 = Q(4) and $\lim_{x \to 4} \frac{P(x)}{Q(x)} = 1$

(c) Polynomials P(x) and Q(x) such that P(4) = 0 = Q(4) and $\lim_{x \to 4} \frac{P(x)}{Q(x)} = -2$

(d) Polynomials P(x) and Q(x) such that P(4) = 0 = Q(4) and $\lim_{x \to 4} \frac{P(x)}{Q(x)} = \infty$

(e) Polynomials P(x) and Q(x) such that P(4) = 0 = Q(4) and $\lim_{x \to 4} \frac{P(x)}{Q(x)}$ does not exist.

Students had studied limits of rational functions in class and had been assigned several problems involving the calculation of such limits for given functions. Question A aimed to develop students' understanding of the limits of rational functions. In the situations presented in the questions (that is, where the limits of both the numerator and the denominator are zero), students often instinctively feel that the limit of the rational function must then be 0/0 and so does not exist. The task designer hoped that by attempting this task, students would gain an understanding of the different outcomes that can arise from this situation and an appreciation of the reasons for these different outcomes. In addition, it was envisaged that students' facility with constructing polynomials with certain properties would be enhanced and that students' skills for dealing with unfamiliar tasks would be developed.

We will report on the students' characterisations of example generation exercises, the strategies they adopted in an effort to solve such problems, and what they feel they learnt in the process.

Students' characterisations of these tasks

Three out of five students interviewed spontaneously characterised the example generating problems as requiring them to work backwards.

> Student A: The backwards ones?...she usually gives us the answers and asks us to find the question….She gives you the answer to the limit and asks you to find the function of the limit. It's the opposite way.

> Student T: It's kind of working backwards from what I'm used to. Normally I'm given the function, I have to work from the function, not the other way around.

> Student P: You had to actually work not only backwards but forwards in answering the questions.

Students also characterised them as being unfamiliar because they had not been told in advance how to approach this type of problem or because they are different to the questions in the textbooks.

> Student C: I might see a question that we hadn't actually covered in our lecture slides. I'd be a bit confused why we got it.
>
> Student P: I find that there was a difference between [the lecturer's] questions and let's say exercises in the book.

Students' approaches to these tasks

Most of the interviewees described struggling with Question A initially, reporting that they did not know how to get started.

> Student E: At first I found it difficult to understand how to go about approaching that.

Some students reported using trial and error:

> Student P: The way I started when I was working through it was just by trial and error

Others appeared to be trying to create an algorithm for themselves:

> Student C: It's just -- trying to get that first step. Once you've done it you just kind of follow your steps as long as it takes to get your answer
>
> Student E: After working out how to do it, it certainly helped to cement it in my mind in what way to approach that question. It gave me a good understanding of how to come up with these functions

What students learned/gained from these tasks

Most students appeared to appreciate the benefits of unfamiliar tasks. They reported improvements in their conceptual understanding and acknowledged the quality of example generation tasks as learning tools.

> Student E: But you really have to think more about things and understand the concepts and the different - ahm - possible solutions that may be there and why one solution isn't going to work……That would be more to cement the whole concept, why it works this way and - you know to cement that into your mind.
>
> Student C: I think this one is more focussed on like why you're using certain things instead of just like how.
>
> Student P: the conceptual ones ah helped you to learn the topic better, because you really know why it works not just how it works.
>
> Student A: I think the backward ones are really good for learning.

Although one student did not see any advantages to the example generation tasks.

> Interviewer: What do you think is the purpose of a question like Question [A]?
>
> Student T: I don't know. I don't think we are ever going to use it in real life.

Some students reported experiencing some satisfaction from completing a somehow more demanding task.

> Student C: Oh ya, it's a good sense of satisfaction.

However, for another the sense of satisfaction seemed to be related to the marks awarded.

> Student T: It has been rewarding to see that I got some marks for it. If I had spent all this hours and I got it wrong it would have been very frustrating.

Concluding Remarks

A preliminary analysis of the interviews supports findings from the literature with respect to the strategies students reported adopting when tackling example generation tasks – for instance, trial and error, and searching for or creating an algorithm as reported by Hazzan and Zazkis (1999).

Most of them recognised the purposes of the tasks and acknowledged the effectiveness of example generation problems as learning tools, in particular, for promoting conceptual understanding. Students were assigned unfamiliar tasks in order to enhance their repertoire of mathematical skills. Enriching the range of activities available to students when tackling or approaching a mathematical problem may be the first step for them towards becoming mathematical experts. This is a view supported by Dreyfus (1991). The interviewees' reflections on their own learning through these unfamiliar tasks were generally positive.

The students' characterisations of example generation tasks as "the backwards ones" surprised us initially. Perhaps the association of example generating problems with backwards motions may be seen as an endorsement of claims that mathematics learning at post-primary level in Ireland focusses on procedural step-by-step learning, in which the order of steps plays a significant role. The students' initial reaction here was to relate a new task with those familiar to them. However, the act of reversing a familiar procedure, or 'undoing', can itself provide a valuable learning opportunity leading to a creative range of possibilities which students can explore, and may generate interest and surprise (Mason and Johnston-Wilder 2004). In fact, Mason and Johnston-Wilder maintain that the "point of working through examples is not only to practice the technique but also to obtain insight into the underlying structure" (45). This is the reason that they advocate learners not only work with given examples but also generate their own.

Acknowledgement

The authors would like to acknowledge the support of a NAIRTL grant for the Task Design Network project.

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