Role of example generation in implicit and explicit conjecturing tasks

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Conjecturing is crucial to mathematics and an activity in which it is believed mathematics learners of all ages should engage. It has been found that mathematicians productively generate examples when they are formulating conjectures. In this paper we explore whether this is also the case for undergraduate non-specialist mathematics students by means of an instrumental case study. We label conjecturing tasks as either explicit or implicit to distinguish between tasks which explicitly ask students to make a conjecture and those in which conjecturing is evoked implicitly, and we discuss the benefit of Comprehensive Example Generation (by which an example set is generated sequentially and systematically) in the context of such conjecturing tasks. The consequences of using a digital environment for such tasks are also discussed.

Keywords: conjecturing, example generation, task design, learning of calculus, students' practices at university level.

INTRODUCTION

Conjecturing and proving are fundamental practices of mathematicians. Indeed Bass (2015) describes most mathematical research as progressing from exploration and discovery, through conjecturing to formal proof. Yet conjecturing tends to be neglected in mathematics teaching and learning (Belnap & Parrott, 2013). This is despite the fact that many mathematics educators and researchers (e.g. Rasmussen et al., 2005) agree that learners of mathematics should be active participants in mathematical practices – including conjecturing, generalising and proving. In fact, ensuring learners' activities are synonymous with the practices of mathematicians is now a goal of national mathematics standards in many countries (Canadas et al., 2007).

Belnap and Parrott (2013) assert that the activity of conjecturing is crucial in mathematics but yet is accessible to learners due to its speculative nature and does not require the same rigour and precision as deductive work. Harel et al. (2022) agree that students should explore mathematical situations before being asked to construct proofs and contend that dynamic geometry environments have potential for *engaging* students in the process of conjecturing. Such environments allow students to focus on the core aspects of the mathematical phenomenon they are observing and draw them into a sense-making process. Advantages of using technology are also described by Breda and Dos Santos (2016) who explain how it allows information to be gathered and processed quickly and removes the burden of computation from the student, affording greater opportunities for experimentation and exploration.

Mathematicians use examples regularly and in several ways - for instance, to help them understand a statement or definition, refute a statement or generate an argument (Alcock, 2004). In particular, the contribution of example generation as an approach to a mathematician's fundamental activity of proof production has been acknowledged. Yet, despite the obvious role played by conjecture formulation in proving, Lynch et al. (2022) contend that little is known about the interplay between example exploration and conjecture formation; not alone for mathematicians but also for learners of mathematics. In fact, Furinghetti, Morselli and Antonini (2011) have cautioned that a focus on examples may "make students stick to the explorative stage and inhibit the need for generalization" (p.219) and that students often consider that checking examples constitutes a means of proof. The aim of this paper is two-fold. Firstly to distinguish between two types of conjecturing tasks for non-specialist undergraduate mathematics students: one in which students are explicitly asked to make a general conjecture, following their exploration of specific cases or examples; the other in which students are asked to generate examples of different phenomena but are expected to make a general conjecture as a result. We label the former as *explicit* conjecturing tasks and the latter as *implicit*. We present an example of each type of task (designed by the authors and used in first year Calculus modules) and the hypothetical learning process associated to each of them. Secondly, we put forward some evidence of the affordances for conjecturing provided by example generation tasks. In particular, we explore the research question: what is the role of example generation in facilitating the formulation of mathematical conjectures for undergraduate non-specialist mathematics students?

THEORETICAL FRAMEWORKS

Canadas et al. (2007) explain how different problems lead to different types of conjectures and propose a classification of conjecturing activity. They characterise conjectures as belonging to one of the following types: 1. Empirical induction from a finite number of cases; 2. Empirical induction from dynamic cases; 3. Analogy; 4. Abduction; 5. Perceptually based conjecturing. They identify the 'stages' of conjecturing associated with each type (e.g. observing cases, validating the conjecture) and explore how the context of a problem can encourage or discourage different types of conjecturing, cautioning that problem selection is important if specific types of conjectures are desired. We note that many of the problems described in Canadas et al. (2007) illustrating the different types of conjecturing discussed there are what we have termed 'implicit' conjecturing tasks.

The stages of conjecturing identified by Canadas et al. (2007) can be viewed as a 'hypothetical learning process' which together with learning goals and learning activities form a 'hypothetical learning trajectory' (HLT) for a student. Simon and Tzur (2004) recommend the use of HLTs in task design to ensure that sufficient thought is given to the development of student thinking through engagement with a task. This recommendation has been taken up by a number of researchers (e.g., Stylianides & Stylianides, 2009; Breen et al., 2019) to tie theory to practice and examine whether the intended goals of an instructional task or sequence of tasks have been achieved.

Lynch, Lockwood and Ellis (2022) focus on mathematicians' practice of generating examples when formulating new conjectures and introduce the term *Comprehensive Example Generation* (CEG) to describe the act of systematically and sequentially generating a data set. Two conditions must be satisfied for CEG; firstly, that the intention in generating the set is to reveal a structure or pattern, and secondly, that the systematic process used would reveal all examples of the phenomenon of interest if continued indefinitely. In interviews with thirteen mathematicians, Lynch et al. (2007) found that CEG had particular affordances for conjecturing activity.

SAMPLE TASKS

We describe two tasks here which encourage students to make conjectures following example generation, one explicit and one implicit. Both were designed by two of the authors as part of a previous research project. The aim was to develop tasks which would introduce undergraduate students to the (previously unfamiliar) habits of mind of mathematicians and provide them with opportunities to develop their mathematical thinking skills and understanding. The habits of mind on which we focussed included example generation, conjecturing and generalising (Breen and O'Shea, 2019). We use the types of conjecturing tasks identified by Canadas et al. (2007) to categorise the two tasks, and we outline the activity and learning we expect from students when engaging with each task.

The Subset Task (Explicit Conjecturing Task)

Students are asked to find examples of a phenomenon, in this case a set with exactly k subsets for different values of k, and then to make a conjecture about how many subsets a set with n elements has. By constructing examples it is hoped that the existence *and non-existence* of an example can give students opportunities to make conjectures.

A. Can a set with exactly 2 subsets be found? Explain.
B. Can a set with exactly 3 subsets be found? Explain.
C. Can a set with exactly 4 subsets be found? Explain.
D. Can a set with exactly 5 subsets be found? Explain.
E. Can a set with exactly 6 subsets be found? Explain.
F. Suppose a set S has n elements. Make a conjecture for the number of subsets that S has.

Figure 1: The Subset Task

<u>Hypothetical Learning Process (HLP)</u>

We expect students to

- determine that a set with one element has 2 subsets and then attempt to construct a set with 3 subsets by looking at a set with two elements,
- realise that a set with two elements has 4 subsets and conclude that it is not possible to have a set with 3 subsets,
- construct a set with three elements, determine it has 8 subsets and realise it is not possible to construct a set with 5 or 6 subsets,
- recognise the 'power of 2' structure in the sequence 2, 4, 8,
- conjecture that the number of subsets of a set of size n is 2^n,

- test their conjecture with n=4 to verify that a set with 4 elements has 16 subsets.

As the students generate this data set systematically and comprehensively, in theory all examples and non-examples would appear in time in line with CEG theory (Lynch et al., 2007), lending strength to the underlying structure observed.

Conjecture type: Empirical induction from a finite number of discrete cases

This task involves 'empirical induction from a finite number of discrete cases' and as such it is a 'Type 1' conjecturing task following Canadas et al (2007). It is expected that students would progress through the stages of a Type 1 task as outlined there, namely: observing cases; organising cases; searching for and predicting patterns; formulating a conjecture; validating the conjecture; generalising the conjecture; justifying the generalisation. These stages, with the omission of the last, align well with the 'steps' outlined above in the Hypothetical Learning Process for the task.

The Asymptotes Task (Implicit Conjecturing Task)

Students were encouraged to use the dynamic geometry software Geogebra to look for examples of different phenomena. They were not explicitly asked to make conjectures.

> Consider the graph of the rational function
>
> $$f(x) = \frac{ax + b}{cx^2 + d}$$
>
> Is it possible to choose values of *a, b, c, d* (between -5 and 5) in order to provide an example of a function of this type such that:
>
> (i) the graph of *f(x)* has no vertical asymptotes;
>
> (ii) the graph of *f(x)* has one vertical asymptote;
>
> (iii) the graph of *f(x)* has more than one vertical asymptote;
>
> (iv) the graph of *f(x)* has no horizontal asymptote.

Figure 2: The Asymptotes Task

The students were provided with a Geogebra applet with sliders; this allowed them to change the values of the coefficients *a, b, c, d,* and observe what effect the changes had on the graph of the rational function. The applet's initial state was to set all of the coefficients *a, b, c, d* to 1 (see Figure 3A). In this configuration, the function has no vertical asymptote and the x-axis is a horizontal asymptote of the graph. Note that the function will only have vertical asymptotes when c and d have different signs. It will have exactly one vertical asymptote in the case when the numerator is a factor of the denominator (for example if a=b=c=1 and d=-1 as in Figure 3B). If c=0 and d is non-zero then the function is linear and has neither vertical nor horizontal asymptotes.

Hypothetical Learning Process (HLP)

We would expect students to

- experiment with the values of all of the coefficients,
- realise that the existence of a vertical asymptote depends on the coefficients of the polynomial in the denominator and so begin by changing the values of *c* and *d*,
- look at the graphs of functions where $c>0$ and $c<0$, then similarly for *d*, and realise that the denominator of *f(x)* only has zeros if *c* and *d* have different signs,
- observe that the *x*-axis is a horizontal asymptote for the function *f(x)* except when $c=0$ and the function is linear.

<u>Conjecturing type: Empirical induction from dynamic cases</u>

This task is close to a Type 2 task as categorised by Canadas et al. (2007), as it affords 'empirical induction from dynamic cases'. The stages in such a task are described as: manipulating a situation dynamically through continuity of cases; observing an invariant property in the situation; formulating a conjecture that the property holds in other cases; validating the conjecture; generalising the conjecture; justifying the generalisation. However, it is noted that not all of the stages necessarily occur with every conjecture. The stages described correspond well to the 'steps' outlined above in the Hypothetical Learning Process for the task, although there are a number of conjectures which can be made in response to the different parts of the task.

CASE STUDY: IMPLICIT CONJECTURING TASK

We present some evidence here that example generation tasks can provide opportunities for students to engage in conjecturing behaviours. We consider this to be an instrumental case study (Stake, 2000) where an instance of a phenomenon is explored in an effort to understand more about the general phenomenon. In our study, the case is the work of two students on an implicit conjecturing task.

Methodology

The second author carried out task-based interviews with four students from an introductory undergraduate calculus module where tasks like the Asymptotes task were assigned. The interviews each lasted for about an hour; during this time the students completed between 4 and 7 tasks and were encouraged to 'think aloud' throughout. Special software was used to record video, audio, the computer screen and any mouse movements. The transcription of the interviews included the audio recording along with a description of what was happening on screen. Two of the students, to whom we have given the pseudonyms Áine and Máire, worked on the Asymptotes task. Their transcripts have been analysed by two of the authors using a deductive approach to apply the theoretical frameworks and conceptualise the data. We sought episodes in the transcripts which provided evidence of the students engaging in CEG, reaching a particular point in the Hypothetical Learning Process or working at a particular stage of the conjecturing process as envisaged by Canadas et al. (2007).

Student Data for the Asymptotes Task

We consider the responses of Áine and Máire on the Asymptotes task. Prior to using the Geogebra applet, these students were given a paper version of the task. Áine gave a correct example (by setting *c*=0) to part (i) but was not able to provide examples for the other parts. For (i), Máire said that it was not possible to have an example of this type, for (ii) she said that *x* (not *c*) and *d* should both be zero, and she was unable to provide examples for parts (iii) and (iv). When using the Geogebra applet, both students started by moving one slider at a time while making sure that the other coefficients were set to 1. We will consider their work on this task individually.

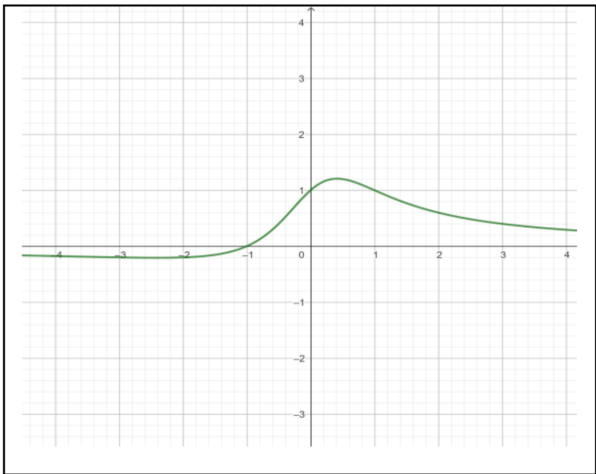
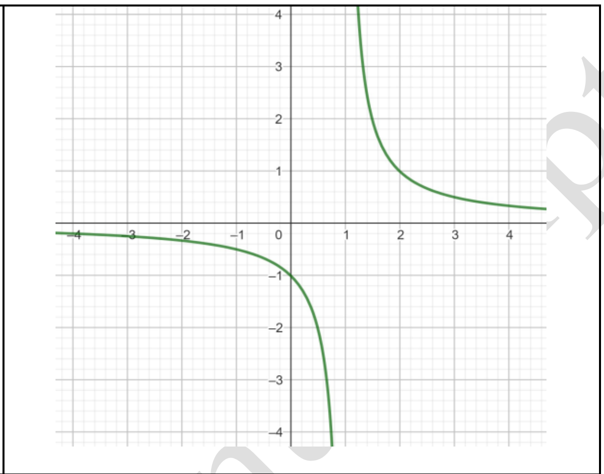
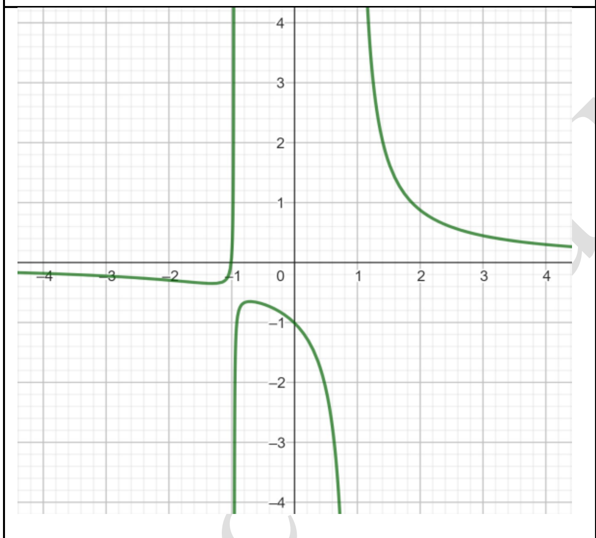
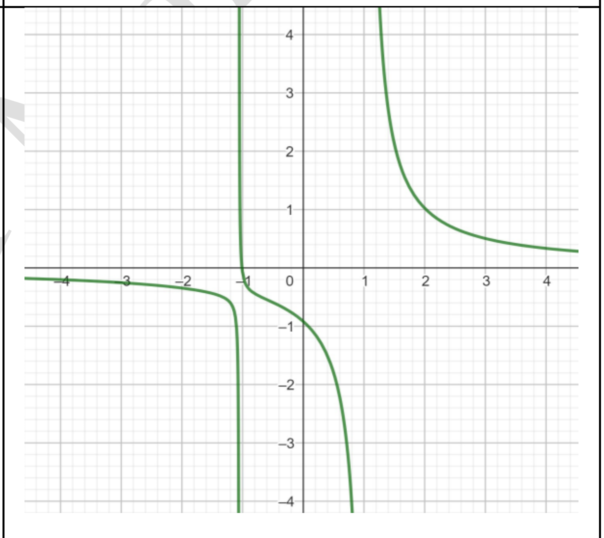
	
A: *a=b=c=d=1*	B: *a=b=c=1 d= -1*
	
C: *a=b=1, c=1.1, d= -1*	D: *a=b=c=1, d= -1.1*

Figure 3: Graph of *y=f(x)* for various values of *a, b, c,* and *d* for the Asymptotes Task

Áine first selects the slider for *d* and changes the value to -1; she correctly observes aloud that the graph has a vertical asymptote (see Figure 3B). She changes the values of *d* to values ranging from -0.2 to -2.2 (the graph looks like the one in Figure 3C for values of *d* in (-1, 0) and like the graph in Figure 3D for values of *d* in (-5, -1)). Áine seems to notice the changes in the graph around *d*=-1, She then moves the value of *d* to values in (1,5) (and sees graphs similar to that in Figure 3A) and says '..higher values of *d* it looks closer to a curve'. At this point she sets *d* back to 1 and moves the slider for *c*. To begin with she looks at positive values of *c*, then she puts *c*=0 and notices that the graph is linear. She moves the slider for *c* back to 1 and then changes the value of *b* first and then *a*. She describes the resulting curves and notes that the *x*-axis is a horizontal asymptote of all of them. She then returns to the starting values of the coefficients and says 'the vertical asymptotes occurred when I change *d* but not when I change other values…ok'. This could be taken as a conjecture. She then tries moving both *c* and *d* from their values of 1. She is able to give correct answers to parts (i)-(iv). She says there will be more than one vertical asymptote for 'different values of *c* and *d* working together'. This seems like a conjecture that the relationship between *c* and *d* is crucial to the existence of vertical asymptotes, but she does not specify a relationship. At this point she seems to have revised her earlier conjecture that the existence of an asymptote depends only on *d*. So Áine has been able to use the task to experiment, to create examples, and to make and revise some conjectures.

Máire begins by changing the values of *a* from 1 to 5 and then down to -5, then returns *a* to 1 and changes *b* in the same way. After both sets of manipulations, she says that these functions have no vertical or horizontal asymptotes. Note that she is correct about the non-existence of vertical asymptotes here but all of these functions have a horizontal asymptote at $y=0$ (see for example Figure 3A). Máire sets *a* and *b* to 1, moves *c* up to 5, and says that there are no asymptotes. She then moves the value of *c* to -5 and says that there are again no asymptotes (which is incorrect as the graph looks like that in Figure 3C reflected in the *x*-axis). She sets *c* to be 0, notices that the graph is now linear and realises why. She changes the value of *c* to be 2 and moves *d* to negative values. This gives a curve similar to the one in Figure 3C. She notices that the *x*-axis is a horizontal asymptote and says that she thinks this function has a vertical asymptote also. Máire is able to find examples of a graph with no vertical asymptote and two vertical asymptotes by changing the values of *c* and *d* at the same time. She says 'the more negative *d* gets the more vertical asymptotes we have' which is not true but is a conjecture. Then she moves *d* from -5 to -0.8 and *c* from 1 to 5. She says that if we make *c* positive and *d* negative then we have a horizontal asymptote (this is true and possibly a conjecture but she does not mention the existence of a vertical asymptote here). She finds the example in Figure 3B, and moves *c* to 5 and *d* to -5 which again has one vertical asymptote. Note that the graph has two vertical asymptotes for most functions in this range of values for *c* and *d*. She says that 'we have more than one vertical asymptote if $c<0$, $d<0$ and $c>d$'. This is a conjecture but it is not true. So Máire was able to use the Geogebra task to experiment, to find some examples, and to make conjectures. However, most of her conjectures are not correct.

DISCUSSION

It can be difficult in a course for non-specialist students to find opportunities to engage in authentic conjecturing activities. We have presented two types of example generation tasks here that can lead students to observe patterns and to form their own conjectures. Lynch et al. (2022) have demonstrated how mathematicians use systematic example generation techniques to form conjectures, however little is known about students' tendencies to use CEG. The Subset task was designed to elicit this behaviour. Although we do not present data on that task here, we have included it as an example of a task in which students are *explicitly* invited to engage in conjecturing and by which the process of CEG acts as a means of focussing students' attention on the inherent structure in a set of examples, thereby facilitating generalisation and the formulation of a conjecture. In the Subset task, students need not only to find examples, but also to realise that in certain cases this may be impossible – that is, they must combine the information from their examples and, crucially, non-examples in order to observe the expected pattern. In this way, there is greater agency and responsibility given to students in exploring the situation than might be if the task were to ask students simply to find the number of subsets for a set with (i) one, (ii) two, (iii) three elements. In addition, it may be that students realise that they can find a set with two subsets, and one with four subsets but none with three subsets and make a preliminary conjecture that the number of subsets must be even. Realising that there is no set with exactly six subsets would then cause them to refine this conjecture and arrive at a correct supposition for the number of subsets.

In the Asymptotes task, the students had difficulty in achieving CEG. It is possible that the issue was that there is a continuum of examples related to that task, and the students were able to move from one example to another very quickly. In some cases, a tiny change in one of the coefficients led to a significant change in the shape of the graph (compare the graphs in Figure 3B, C and D which arose from changes of 0.1 in the coefficients). Máire, in particular, changed the coefficient values quite quickly and this may have made it difficult for her to detect where important changes took place. Eventually the visualisation was valuable for her, and at the end of the task Máire said 'I didn't realise that the conditions had to be so specific until I actually looked at it …till I looked at it graphically really'. The task was difficult for the two students who attempted it; however, they were successful in generating examples and both conjectured that the relationship between *c* and *d* was important.

Technology helps students with these types of tasks because it allows them to look at a large number of possible examples quickly without the burden of calculation and to focus on the patterns emerging. In our study, both students had more success on the Asymptotes task when using the Geogebra applet than when they attempted the same task on paper. However, it may be that the speed at which students encounter examples when using such software is a problem and they may miss some important features. Harel et al. (2022) report that the facility and immediacy of generating large numbers of examples with technology may actually hinder the process of formulating and refining conjectures. As we saw with Máire and Áine, while the environment aided them in making a number of conjectures, their conjectures were often false, and they generally did not verify and subsequently refute or refine their conjectures. Lynch et al (2022) note that if examples are not generated using a systematic process, the example set might be non-comprehensive and may reveal misleading patterns. Furthermore, they caution that even when a student tests a diverse collection of examples by systematically varying one or more elements, all possible examples of a phenomenon may not be revealed. This appears to be true of Máire and Áine's work on this task.

The Asymptotes task can be categorised as an ‘empirical induction from dynamic cases’ (or Type 2 task) following Canadas et al. (2007). However, the students did not follow the stages outlined there in their approach to it. While they do manipulate the situation dynamically, they often do not observe an invariant property. They make conjectures in some cases, based on the manipulations that they have carried out, but do not attempt to validate or generalise their conjectures. It could be that the focus on giving examples impeded the act of generalisation as was found by Furinghetti, Morselli and Antonini (2011). Alternatively, it could be that the existence of *four* parameters which can be changed makes the task too complex for the students at this point in their learning and hampered their progress through the stages of conjecturing predicted by Canadas et al. (2007). The students in this study also completed simpler tasks using a Geogebra applet where only one parameter was involved. For those tasks, the students did identify the underlying patterns, and the use of the dynamic geometric environment seemed to be positive both from the perspective of engagement and the development of their thinking (Breen et al., 2019.)

While Áine and Máire did not closely follow the learning process hypothesised for the implicit conjecturing (Asymptotes) task, we believe there is a role for such implicit tasks in the curriculum. It may be that students need to become accustomed with CEG through multiple opportunities to complete tasks such as the Subset task in order for systematic and sequential generation of examples to develop as a ‘habit of mind’. However, we note that both students did make conjectures even though this was not explicitly asked for. This gives us confidence that a disposition of conjecturing is being developed. It may be that their learning could have been scaffolded more effectively by adding more structure to the Asymptotes task while retaining its implicit nature.

Implicit and explicit conjecturing tasks have characteristics that can help students develop their mathematical thinking skills. Explicit conjecturing tasks which involve CEG (such as the Subset task) have the potential to provide structure within which students can explore possibilities and make conjectures, while implicit conjecturing tasks which arise from example generation in a complex situation can encourage students to make conjectures naturally. While the two conjecturing tasks presented here were used in first-year undergraduate Calculus modules, we have designed and used similar tasks in other areas (e.g., Analysis, Number Theory) and with other undergraduate students and have found them to be equally useful in those contexts to engage students in conjecturing.

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