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Stochastic equations for interacting particle systems with continuous spins

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Abstract. We study a general class of interacting particle systems over a countable state space V where on each site $x \in V$, the particle mass $\eta(x) \geq 0$ follows a stochastic differential equation. We construct the corresponding Markovian dynamics in terms of strong solutions to an infinite coupled system of stochastic differential equations and prove a comparison principle with respect to the initial configuration as well as the drift of the process. Using this comparison principle, we provide sufficient conditions for the existence and uniqueness of an invariant measure in the subcritical regime and prove convergence of the transition probabilities in the Wasserstein-1-distance. Finally, we establish a linear growth theorem for sublinear drifts showing that the spatial spread is at most linear in time. Our results cover a large class of finite and infinite branching particle systems with interactions among different sites.

Résumé. Nous étudions une classe générale de systèmes de particules en interaction sur un espace d'états dénombrable V où, sur chaque site $x \in V$, la masse de la particule $\eta(x) \geq 0$ suit une équation différentielle stochastique. Nous construisons la dynamique markovienne correspondante en termes de solutions fortes d'un système couplé infini d'équations différentielles stochastiques et prouvons un principe de comparaison en ce qui concerne la configuration initiale ainsi que la dérive du processus. En utilisant ce principe de comparaison, nous fournissons des conditions suffisantes pour l'existence et l'unicité d'une mesure invariante dans le régime sous-critique et prouvons la convergence des probabilités de transition dans la distance de Wasserstein-1. Enfin, nous établissons un théorème de croissance linéaire pour les dérives sous-linéaires montrant que la propagation spatiale est au plus linéaire en temps. Nos résultats couvrent une large classe de systèmes de particules à ramifications finies et infinies avec des interactions entre différents sites.

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1. Introduction

We consider a continuous-state branching process over a countable space V. For a fixed weight function $v : V \longrightarrow (0,\infty)$, we define the space of tempered configurations over V via

$$\mathcal{X} = \left\{ \eta = (\eta(x))_{x \in V} \mid \eta(x) \in \mathbb{R}_+ \ \forall x \in V \text{ and } \sum_{x \in V} \eta(x) v(x) < \infty \right\}. \tag{1}$$

Then on $\mathcal{X}$ we consider the norm restricted onto the cone $\mathcal{X}$ given by

$$\|\xi\| := \sum_{x \in V} v(x) |\xi(x)|, \quad \xi \in \mathcal{X}.$$

The distance between $\eta, \xi \in \mathcal{X}$ is then given by

$$d(\eta, \xi) = \|\eta - \xi\| = \sum_{x \in V} v(x) |\eta(x) - \xi(x)|.$$

We equip $\mathcal{X}$ with the corresponding Borel σ-algebra. For a given configuration $\eta = (\eta(x))_{x\in V} \in \mathcal{X}$ the number $\eta(x) \geq 0$ describes the mass of particles at the site $x \in V$. By $0 \in \mathcal{X}$ we denote the empty configuration $\eta(x) = 0$ for all $x \in V$. The weight function v allows for a flexible treatment of finite and infinite particle systems. Indeed, if $\inf_{x\in V} v(x) > 0$, then elements in $\mathcal{X}$ are necessarily summable sequences that correspond to finite particle configurations. Conversely, if $v(x)$ has sufficient decay at "infinity", then $\mathcal{X}$ may contain sequences that are not summable corresponding to infinite tempered configurations.

In this work, we study particle dynamics on $\mathcal{X}$ where at each moment $t \geq 0$ and each $x \in V$, the value of the process $\eta_t(x)$ represents the mass at location x at time t. This mass follows the system of stochastic equations

$$
\begin{aligned}
(2) \qquad \eta_t(x) &= \eta_0(x) + \int_0^t B(x,\eta_s)ds + \int_0^t \sqrt{2c(x,\eta_s(x))}dW_s(x) \\
&\quad + \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x)\mathbb{1}_{\{u\leq g(x,\eta_{s-}(x))\}} \widetilde{N}_x(ds,d\nu,du) \\
&\quad + \sum_{y\in V\setminus\{x\}} \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x)\mathbb{1}_{\{u\leq g(y,\eta_{s-}(y))\}} N_y(ds,d\nu,du) \\
&\quad + \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x)\mathbb{1}_{\{u\leq \rho(x,\eta_{s-},\nu)\}} M(ds,d\nu,du)
\end{aligned}
$$

where $B(x,\cdot) : \mathcal{X} \longrightarrow \mathbb{R}$, $c(x,\cdot), g(x,\cdot) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $c(x,0) = g(x,0) = 0$, and $\rho : V \times \mathcal{X} \times \mathcal{X}\setminus\{0\} \longrightarrow \mathbb{R}_+$ are measurable functions. All these parameters are supposed to satisfy the additional conditions specified below. The noise terms are defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with the usual conditions and satisfy the following assumptions:

- (N1) $(W_t(x))_{t\geq 0,\ x\in V}$ are mutually independent one-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motions.
- (N2) $(N_y(ds,d\nu,du))_{y\in V}$ are mutually independent $(\mathcal{F}_t)_{t\geq 0}$-Poisson random measures on $\mathbb{R}_+ \times \mathcal{X}\setminus\{0\} \times \mathbb{R}_+$ with compensator $\widehat{N}_y(ds,d\nu,du) = dsH_1(y,d\nu)du$, where $H_1(y,d\nu)$ is, for each $y \in V$, a sigma-finite measure on $\mathcal{X}\setminus\{0\}$.
- (N3) $M(ds,d\nu,du)$ is an $(\mathcal{F}_t)_{t\geq 0}$-Poisson random measure on $\mathbb{R}_+ \times \mathcal{X}\setminus\{0\} \times \mathbb{R}_+$ with compensator $M(ds,d\nu,du) = dsH_2(d\nu)du$, where $H_2(d\nu)$ is a sigma-finite measure on $\mathcal{X}\setminus\{0\}$.
- (N4) The noise terms $(W_t(x))_{t\geq 0,\ x\in V}$, $(N_y(ds,d\nu,du))_{y\in V}$, $M(ds,d\nu,du)$ are independent.

Finally, we let $\widetilde{N}_x = N_x - \widehat{N}_x$ denote the compensated Poisson random measure. We employ the following standard definition of weak and strong existence. A strong solution of (2) is an $(\mathcal{F}_t)_{t\geq 0}$-adapted cadlag process $(\eta_t)_{t\geq 0} \subset \mathcal{X}$ such that (2) holds a.s. for each $x \in V$ and $t \geq 0$. A weak solution consists of a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, an $(\mathcal{F}_t)_{t\geq 0}$-adapted cadlag process $(\eta_t)_{t\geq 0} \subset \mathcal{X}$ and noise terms (N1) – (N4) such that (2) holds a.s. for each $x \in V$ and $t \geq 0$. In this definition, we implicitly assume that all integrals in (2) are well-defined.

The system of stochastic equations (2) contains finite and infinite dimensional models studied in the literature. In the finite-dimensional case, $V = \{1, \dots, m\}$ with $v(x) = 1$, we have $\mathcal{X} = \mathbb{R}_+^m$. If the coefficients $B(x,\eta), c(x,t), g(x,t)$ are affine linear in η and t, and $\rho = 1$, then (2) reduces to multi-type continuous-state branching processes with immigration as constructed in [4], see also e.g. [24, Chapter 3] for the one-dimensional case $m = 1$. Moreover, for general B, c, g and $V = \{1\}$ equation (2) reduces to nonlinear continuous-state branching processes with immigration studied in [14], [22], [31, Chapter 8] and [11]. The multidimensional CBI process is also mentioned in [24, Example 9.1]. Thus, the stochastic particle system studied in this work provides an infinite-dimensional extension of multi-type CBI processes and their non-linear analogues. More generally, for infinite state spaces V, our model covers a wide class of interacting particle systems including, e.g., multi-type branching systems [9, 32], population models with interactions [16], systems of particles driven by α-stable noises [35], branching random walks with discrete state space [33]. For other related literature see [8].

Here and below we write, for $\eta, \xi \in \mathcal{X}$, $\eta \leq \xi$ if $\eta(x) \leq \xi(x)$ holds for all $x \in V$. We impose the following conditions on the coefficients of (2):

- (A1) The drift coefficient $B(x,\eta)$ has the form $B(x,\eta) = B_0(x,\eta) - B_1(x,\eta(x))$ where $B_0(x,\cdot) : \mathcal{X} \longrightarrow \mathbb{R}_+$ and $B_1(x,\cdot) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are measurable mappings for each $x \in V$. For each $R > 0$ there exists a constant $C_1(R) \geq 0$ such that

$$
\|B_0(\cdot,\eta) - B_0(\cdot,\xi)\| \leq C_1(R)\|\eta - \xi\|
$$

holds for all $\eta, \xi \in \mathcal{X}$ with $\|\eta\|, \|\xi\| \leq R$. The function $\mathbb{R}_+ \ni t \longmapsto B_1(x,t)$ is continuous, non-decreasing, and $B_1(x,0) = 0$ holds for each $x \in V$. Finally, for all $\eta, \xi \in \mathcal{X}$ satisfying $\eta \leq \xi$, it holds that

$$B_0(x,\eta) \leq B_0(x,\xi), \qquad \forall x \in V.$$

(A2) For each $x \in V$ there exists a constant $C_2(x) \geq 0$ such that

$$|c(x,t) - c(x,s)| \leq C_2(x)|t-s|$$

holds for $t, s \in \mathbb{R}_+$, and

$$\sum_{x \in V} v(x) C_2(x) < \infty.$$

Furthermore $c(x,0) = 0$ for $x \in V$.

(A3) For each $x \in V$ there exists a constant $C_3(x) \geq 0$ such that

$$|g(x,t) - g(x,s)| \leq C_3(x)|t-s|, \qquad t,s \geq 0.$$

The function $\mathbb{R}_+ \ni t \longmapsto g(x,t)$ is non-decreasing for each $x \in V$, and one has $g(x,0) = 0$ for each $x \in V$.

(A4) It holds that

$$\sum_{x \in V} v(x) C_3(x) \int_{\{\|\nu\| \leq 1\} \setminus \{0\}} \nu(x)^2 H_1(x, d\nu) < \infty,$$

and, there exists a constant $C_4 \geq 0$ such that

$$C_3(x) \int_{\{\|\nu\| > 1\}} \nu(x) H_1(x, d\nu) \leq C_4$$

and

$$C_3(x) \int_{\mathcal{X} \setminus \{0\}} \sum_{y \in V \setminus \{x\}} v(y) \nu(y) H_1(x, d\nu) \leq C_4 v(x)$$

hold for all $x \in V$.

(A5) For each $R > 0$ there exists a constant $C_5(R) \geq 0$ such that

$$\int_{\mathcal{X} \setminus \{0\}} \sum_{x \in V} v(x) \nu(x) |\rho(x,\eta,\nu) - \rho(x,\xi,\nu)| H_2(d\nu) \leq C_5(R) \|\eta - \xi\|$$

holds for all $\eta, \xi \in \mathcal{X}$ with $\|\eta\|, \|\xi\| \leq R$. For all $\eta, \xi \in \mathcal{X}$ satisfying $\eta \leq \xi$, it holds that

$$\rho(x,\eta,\nu) \leq \rho(x,\xi,\nu), \qquad \forall x \in V, \;\; \nu \in \mathcal{X} \setminus \{0\}.$$

(A6) There exists a constant $C_6 \geq 0$ such that

$$\|B_0(\cdot,\eta)\| + \int_{\mathcal{X} \setminus \{0\}} \sum_{x \in V} v(x) \nu(x) \rho(x,\eta,\nu) H_2(d\nu) \leq C_6 (1 + \|\eta\|).$$

We say that the tuple $(B, B_0, B_1, c, g, \rho)$ is $C_{\overline{1,6}}$-admissible if conditions (A1)–(A6) are satisfied with given $C_1, \dots, C_6$. Sufficient conditions for these assumptions and particular examples are discussed in the next section. Under this general set of conditions, we derive the following existence and uniqueness result of strong solutions and the comparison property of solutions.

Theorem 1.1. *Suppose that conditions (A1) – (A6) are satisfied. Then for each $\mathcal{F}_0$-measurable random variable η_0 with $\mathbb{E}[\|\eta_0\|] < \infty$ there exists a unique strong solution $(\eta_t)_{t \geq 0}$ in $\mathcal{X}$ of (2). Moreover, there exists a constant $C > 0$ independent of η_0 such that $\mathbb{E}[\|\eta_t\|] \leq (1 + \mathbb{E}[\|\eta_0\|]) e^{Ct}$ for $t \geq 0$. Finally, for any $\eta_0, \xi_0 \in \mathcal{X}$ with $\mathbb{E}[\|\eta_0\|]$, $\mathbb{E}[\|\xi_0\|] < \infty$ let $(\eta_t)_{t \geq 0}, (\xi_t)_{t \geq 0}$ be the unique strong solutions of (2). If $\mathbb{P}[\xi_0(x) \leq \eta_0(x), \;\; \forall x \in V] = 1$, then*

$$\mathbb{P}[\xi_t \leq \eta_t, \; \forall t \geq 0] = 1.$$

This theorem will be deduced from the results of Sections 3 - 6. Indeed, in Section 3, we prove the non-explosion and first moment bound for any solution of (2). In Section 4, we establish the pathwise uniqueness of solutions under slightly weaker conditions than (A1) – (A6), while the comparison principle is derived in Section 5. To prove these results, we provide an infinite dimensional extension of the classical Yamada-Watanabe theorem, see also [4, 12, 14, 28] for some finite-dimensional results in this direction. Finally, in Section 6, we prove the weak existence of solutions of (2) using finite-dimensional approximations combined with the comparison principle. Combining all these results gives, in view of the Yamada–Watanabe–Engelbert theorem (see [18]), the strong existence of a unique solution of (2).

The space $\mathcal{X}$ introduced in (1) can be seen as a non-negative cone in an L^1-type space. The space of configurations summable with respect to given weights is a natural choice for a state space in constructing interacting particle systems. This choice goes back at least to Liggett and Spitzer [26] and Andjel [2]. The construction of the stochastic particle systems as solutions to stochastic equations driven by Poisson point processes is not uncommon, as it was used in [6, 15, 27] for the study of birth-and-death processes with an infinite number of particles. Such stochastic equations can be seen as a natural development of the graphical construction for classical interacting particle systems such as the contact process or the voter model [25].

As an application of this construction and the comparison principle with respect to the initial conditions, we prove under a suitable subcriticality condition on the drift the existence and uniqueness of and convergence towards the invariant measure in the Wasserstein distance. Let $\mathcal{P}(\mathcal{X})$ be the space of all Borel probability measures over $\mathcal{X}$ and let $\mathcal{P}_1(\mathcal{X})$ be the subspace of measures with finite first moment, i.e. $\int_{\mathcal{X}} \|\nu\| \varrho(d\nu) < \infty$. Let

$$p_t(\eta, d\xi) = \mathbb{P}[\eta_t \in d\xi \mid \eta_0 = \eta] \tag{3}$$

be the transition probabilities of the process $(\eta_t)_{t\geq 0}$ obtained from (2). A general version of Yamada–Watanabe theorem and Theorem 1.1 imply that the strong solution is unique in law ([18, Theorem 3.14]); therefore p_t in (3) is well defined. Define the semigroup $(P_t^*)_{t\geq 0}$ by $P_t^*\rho = \int_{\mathcal{X}} p_t(\eta, \cdot)\rho(d\eta)$, where $\rho \in \mathcal{P}_1(\mathcal{X})$. Recall that $\pi \in \mathcal{P}(\mathcal{X})$ is called *invariant measure* if $P_t^*\pi = \pi$ for $t > 0$. The Wasserstein-1 distance is defined by

$$W_1(\varrho, \widetilde{\varrho}) = \inf_{G \in \mathcal{H}(\varrho, \widetilde{\varrho})} \left\{ \int_{\mathcal{X}\times\mathcal{X}} \|\eta - \xi\| G(d\eta, d\xi) \right\}$$

where $\mathcal{H}(\varrho, \widetilde{\varrho})$ denotes the set of all couplings of $(\varrho, \widetilde{\varrho})$ on the product space $\mathcal{X} \times \mathcal{X}$. Define the effective drift

$$\widetilde{B}(x,\eta) = B(x,\eta) + \sum_{y \in V\setminus\{x\}} g(y, \eta(y)) \int_{\mathcal{X}\setminus\{0\}} \nu(x) H_1(y, d\nu) + \int_{\mathcal{X}\setminus\{0\}} \nu(x)\rho(x,\eta,\nu) H_2(d\nu). \tag{4}$$

Note that this function is well-defined due to conditions (A4) and (A6).

Theorem 1.2. *Suppose that conditions (A1) – (A6) are satisfied and that the constants from conditions (A1) and (A5) satisfy* $\sup_{R>0}(C_1(R) + C_5(R)) < \infty$. *Assume additionally that there exists* $A > 0$ *such that*

$$\sum_{x\in V} v(x) \left(\widetilde{B}(x,\eta) - \widetilde{B}(x,\xi) \right) \leq -A\|\eta - \xi\| \tag{5}$$

holds for all $\eta, \xi \in \mathcal{X}$ *with* $\xi \leq \eta$. *Then for any* $\varrho, \widetilde{\varrho} \in \mathcal{P}_1(\mathcal{X})$ *one has*

$$W_1(P_t^*\varrho, P_t^*\widetilde{\varrho}) \leq e^{-At} W_1(\varrho, \widetilde{\varrho}), \qquad t \geq 0. \tag{6}$$

In particular, there exists a unique invariant measure $\pi \in \mathcal{P}_1(\mathcal{X})$ *and it holds that*

$$W_1(P_t^*\varrho, \pi) \leq e^{-At} W_1(\varrho, \pi), \qquad t \geq 0. \tag{7}$$

This theorem extends the finite dimensional results [11, 13] to the infinite dimensional setting, but also complements the results discussed in [9, 23, 24] to the case of branching processes with interactions (competition). While the comparison principle remains the key tool to derive stability estimates in the L^1-norm when working with infinite-dimensional settings, additional conditions are required to control the states for all sites $x \in V$. The latter are reflected by the assumption $\sup_{R>0}(C_1(R) + C_5(R)) < \infty$. Assumption (5) is motivated by the one-dimensional case studied in [11] and can be viewed as a subcritical (or strong dissipativity) condition on the drift, i.e., the drift needs to be sufficiently negative. Without immigration, such a condition implies that the empty configuration δ_0 gives the invariant measure. In the presence of non-trivial immigration, the invariant measure is necessarily non-trivial as well.

It is worth mentioning that such a condition is certainly not optimal for branching systems with interactions, as recently was demonstrated in [21] for the one-dimensional case. However, our conditions are relatively simple to verify and allow for a simple illustrative proof. The extension of [21] to our infinite dimensional case is yet an open problem from the literature. Finally, our subcriticality condition (5) also rules out the possibility of multiple invariant measures, a fact natural for certain infinite particle systems as studied in [16]. An extension of our results to such types of settings is, however, beyond the scope of this work.

In the last part of this work, we investigate the growth of a finite particle system without immigration (that is, $H_2(d\nu) = 0$) when started at a single point. We demonstrate that whenever the effective drift $\widetilde{B}$ defined in (4) is sublinear, the spatial spread of the process is at most linear in time. Also, we provide a super-exponential bound in the process's 'large deviations' region. For both results, we suppose that V is the vertex set of an infinite connected graph $G = (V, E)$ of bounded degree. We let $\mathrm{dist}(z, z')$ be the graph distance for $z, z' \in V$, and denote by $\mathbb{B}(x_0, r) \subset V$ the closed ball of radius r with respect to this graph distance.

Let us underline that, unlike particle systems with discrete states, determining what qualifies as an 'occupied site' for a continuous-state process is not straightforward since in many non-degenerate cases we have for $t > 0$

$$\mathbb{P}\Big(\{\eta_t(x) = 0 \text{ for all } x \in V\} \cup \{\eta_t(x) > 0 \text{ for all } x \in V\}\Big) = 1,$$

that is, with probability one, either all sites have a positive mass, or the process has gone extinct, i.e. the total mass is zero. Below we interpret a site as occupied if the particle mass is larger than a given threshold $\varepsilon > 0$.

Theorem 1.3. *Suppose conditions (A1) – (A6) are satisfied with $H_2(d\nu) = 0$ and weight function v. Let $x_0 \in V$ and let $(\eta_t)_{t\geq 0}$ be the unique solution of (2) with initial condition $\eta_0(x) = \mathbb{1}_{\{x=x_0\}}$. Assume that there exists bounded $b : V \times V \longrightarrow \mathbb{R}_+$ such that, for all $x \in V$ and $\eta \in \mathcal{X}$, one has*

$$\widetilde{B}(x, \eta) \leq \sum_{y\in V} b(x, y)\eta(y), \tag{8}$$

there exists $R \in \mathbb{N}$ such that $b(x, y) = 0$ holds for $x, y \in V$ satisfying $\mathrm{dist}(x, y) > R$, and the weight function v satisfies

$$\sup_{\mathrm{dist}(x,y)\leq R} \frac{v(y)}{v(x)} < \infty. \tag{9}$$

Then there exist constants $C, c, m > 0$ such that, for all $x \in V$ and $t \geq 0$, one has

$$\mathbb{E}\left[\sup_{r\in[0,t]} \eta_r(x)\right] \leq C \exp\left[-c\, \mathrm{dist}(x_0, x) \ln(\mathrm{dist}(x_0, x)) + mt\right]. \tag{10}$$

Moreover, for any $\varepsilon > 0$ we find $\mathrm{C}_l > 0$ and a random time $t_0 > 0$ such that a.s.

$$\left\{z \in V : \sup_{r\in[0,t]} \eta_r(z) \geq \varepsilon\right\} \subset \mathbb{B}(x_0, \mathrm{C}_l t), \quad t \geq t_0. \tag{11}$$

The constant C_l may depend only on ε and the parameters of the process.

The assumption on the effective drift $\widetilde{B}$ allows us to compare the process with a simpler process that consists of a linear drift plus a martingale part. The condition on b essentially states that the branching mechanism of this process has finite range. Note that (9) implies that v satisfies with $e^{\kappa} = \sup_{\mathrm{dist}(x,y)\leq R} \frac{v(y)}{v(x)}$ the growth bound

$$v(x_0)e^{-\kappa \mathrm{dist}(x_0,x)} \leq v(x) \leq v(x_0)e^{\kappa \mathrm{dist}(x_0,x)}, \qquad x \in V. \tag{12}$$

The proof of Theorem (1.3) is given in the last section of this work. It relies on the comparison principle combined with heat kernel estimates of a random walk on the graph G. Namely, using the comparison principle combined with the linear growth condition on the effective drift, we obtain a bound of the form $\eta_t \leq \zeta_t$, where ζ_t has a constant drift. It is easy to see that η_t satisfies the assertion whenever the larger process ζ_t satisfies it. To prove that ζ_t satisfies the above theorem, we use an auxiliary graph to apply known heat kernel estimates from [7, Corollary 12].

This work is organized as follows. Section 2 discusses particular examples of (2) and further elaborates on related literature. Section 3 is devoted to the non-explosion and first moment bound on solutions of (2). Pathwise uniqueness is established in Section 4, while the comparison principle is proven in Section 5. Section 6 contains the proof of the weak existence of solutions of (2) while the proof of Theorem 1.2 is proven in Section 7. Finally, the linear spread, that is, Theorem 1.3, is proven in Section 8. A few minor technical results are given in the appendix.

2. Sufficient conditions and examples

As a first step, we state a proposition that allows us to verify conditions (A1) – (A6) in a general framework and hence serves as a toolbox for particular examples.

Proposition 2.1. *Let $v : V \longrightarrow (0,\infty)$ and suppose that*

(i) The drift $B : V \times \mathcal{X} \longrightarrow \mathbb{R}$ is given by

$$B(x,\eta) = \left(b(x) + \sum_{y\in V} a(x,y)\eta(y) \right) - m(x)\eta(x)^{\lambda}$$

where $\lambda \geq 0$, $b, m : V \longrightarrow \mathbb{R}_+$, $\|b\| < \infty$, and $a : V \times V \longrightarrow \mathbb{R}$ is such that $a(x,y) \geq 0$ for $x \neq y$, and there exists $C_1 > 0$ satisfying

$$\sum_{y\in V\setminus\{x\}} v(y)a(x,y) + \mathbb{1}_{\{a(x,x)\geq 0\}} a(x,x)v(x) \leq C_1 v(x), \qquad x \in V. \tag{13}$$

(ii) The diffusion coefficient is given by $c(x,u) = c(x)u$ where, by slight abuse of notation, $c : V \longrightarrow \mathbb{R}_+$ satisfies $\|c\| < \infty$.

(iii) The branching rate is given by $g(x,\eta(x)) = g(x)u$, where $g : V \longrightarrow \mathbb{R}_+$, and $H_1(x,\cdot)$ is a family of σ-finite measures on $\mathcal{X}\setminus\{0\}$ such that

$$\sum_{x\in V} v(x)g(x) \int_{\{\|\nu\|\leq 1\}\setminus\{0\}} \nu(x)^2 H_1(x,d\nu) < \infty$$

and

$$\sup_{x\in V} g(x) \int_{\{\|\nu\|>1\}} \nu(x) H_1(x,d\nu) + \sup_{x\in V} \frac{g(x)}{v(x)} \int_{\mathcal{X}\setminus\{0\}} \sum_{y\in V\setminus\{x\}} v(y)\nu(y) H_1(x,d\nu) < \infty.$$

(iv) The immigration rate-function $\rho : V \times \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}_+$ is given by

$$\rho(x,\eta,\nu) = \sum_{y\in V} \varphi(x,y)\eta(y) + \sum_{y\in V} \psi(x,y)\nu(y)$$

with $\varphi, \psi : V \times V \longrightarrow \mathbb{R}_+$, and $H_2(d\nu)$ is a σ-finite measure on $\mathcal{X}\setminus\{0\}$ such that

$$\sup_{y\in V} \frac{1}{v(y)} \int_{\mathcal{X}\setminus\{0\}} \sum_{x\in V} v(x)\nu(x)\varphi(x,y) H_2(d\nu) < \infty,$$

$$\sum_{y\in V} \int_{\mathcal{X}\setminus\{0\}} \sum_{x\in V} v(x)\nu(x)\psi(x,y)\nu(y) H_2(d\nu) < \infty.$$

Then conditions (A1) – (A6) are satisfied.

Proof. Let us write $B(x,\eta) = B_0(x,\eta) - B_1(x,\eta(x))$ with

$$B_0(x,\eta) = b(x) + \sum_{y\in V\setminus\{x\}} a(x,y)\eta(y) + \mathbb{1}_{\{a(x,x)\geq 0\}} a(x,x)\eta(x),$$

$$B_1(x,\eta(x)) = \mathbb{1}_{\{a(x,x)<0\}} |a(x,x)|\eta(x) + m(x)\eta(x)^{\lambda}.$$

Then B_1 has the desired properties stated in condition (A1) while B_0 satisfies by (13)

$$\|B_0(\cdot,\eta)\| \leq \max\{\|b\|,\ C_1\}(1+\|\eta\|) \ \text{ and } \ \|B_0(\cdot,\eta) - B_0(\cdot,\xi)\| \leq C_1\|\eta-\xi\|, \tag{14}$$

and $B_0(x,\eta) \le B_0(x,\xi)$ for each $x \in V$ whenever $\eta \le \xi$. This shows that (A1) is satisfied with $C_1(R) = C_1$. It is straightforward to verify conditions (A2) with $C_2(x) = c(x)$ and (A3) with $C_3(x) = g(x)$. Condition (A4) follows directly from (iii) with

$$C_4 = \sup_{x\in V} \max\left\{ g(x)\int_{\{\|\nu\|>1\}} \nu(x)H_1(x,d\nu),\ \frac{g(x)}{v(x)}\int_{\mathcal{X}\setminus\{0\}} \sum_{y\in V\setminus\{x\}} v(y)\nu(y)H_1(x,d\nu)\right\}.$$

Concerning assumption (A5) let us note that $|\rho(x,\eta,\nu)-\rho(x,\xi,\nu)| \le \sum_{y\in V}\varphi(x,y)|\eta(y)-\eta(y)|$ and hence

$$\begin{aligned}&\int_{\mathcal{X}\setminus\{0\}} \sum_{x\in V} v(x)\nu(x)|\rho(x,\eta,\nu)-\rho(x,\xi,\nu)|H_2(d\nu)\\ &\le \sum_{y\in V}|\eta(y)-\eta(y)|\int_{\mathcal{X}\setminus\{0\}}\sum_{x\in V} v(x)\nu(x)\varphi(x,y)H_2(d\nu) \le C_5\|\eta-\xi\|\end{aligned}$$

where $C_5 = \sup_{y\in V}\frac{1}{v(y)}\int_{\mathcal{X}\setminus\{0\}}\sum_{x\in V} v(x)\nu(x)\varphi(x,y)H_2(d\nu)$. Finally, condition (A6) is satisfied with

$$\begin{aligned}C_6 = \max\Bigg\{&\|b\|,\ C_1,\ \sup_{y\in V}\frac{1}{v(y)}\int_{\mathcal{X}\setminus\{0\}}\sum_{x\in V} v(x)\nu(x)\varphi(x,y)H_2(d\nu),\\ &\sum_{y\in V}\int_{\mathcal{X}\setminus\{0\}}\sum_{x\in V} v(x)\nu(x)\psi(x,y)\nu(y)H_2(d\nu)\Bigg\}\end{aligned}$$

due to (14) and

$$\begin{aligned}\int_{\mathcal{X}\setminus\{0\}}\sum_{x\in V} v(x)\nu(x)\rho(x,\eta,\nu)H_2(d\nu) &\le \sum_{y\in V}\eta(y)\int_{\mathcal{X}\setminus\{0\}}\sum_{x\in V} v(x)\nu(x)\varphi(x,y)H_2(d\nu)\\ &\quad+\int_{\mathcal{X}\setminus\{0\}}\sum_{x,y\in V} v(x)\nu(x)\psi(x,y)\nu(y)H_2(d\nu)\\ &\le C_6(1+\|\nu\|).\end{aligned}$$

□

Note that the only nonlinear term (in $\eta(x)$) is the mortality $m(x)\eta(x)^\lambda$. In particular, for $\lambda = 2$, we obtain the case of logistic killing, which was studied in [20]. The next remark illustrates specific cases when the inequality (13) is satisfied. It follows the scheme provided in [26].

Remark 2.2. Let $V = \mathbb{Z}^d$ be equipped with the 1-norm $|\cdot|_1$. Let $a : V\times V \longrightarrow \mathbb{R}$ and v be given by one of the following cases:

(i) $v(x) = e^{-\delta|x|_1}$ and $a(x,y) = ce^{-\varepsilon|x-y|_1}$ for $x\ne y$ with $c>0$ and $0<\delta<\varepsilon$,
(ii) $v(x) = e^{-\delta|x|_1}$ and $a(x,y) = c\mathbb{1}_{\{|x-y|_1\le R\}}$ for $x\ne y$ with $c,R,\delta>0$,
(iii) $v(x) = \frac{1}{1+|x|_1^\delta}$ and $a(x,y) = \frac{c}{1+|x-y|_1^\varepsilon}$ for $x\ne y$ with $c>0$ and $d<\delta<\varepsilon$.

Then there exists a constant $C_1>0$ such that (13) holds.

Note that all these examples satisfy condition (9). Under the conditions of the previous proposition, one may check that the effective drift is given by

$$\widetilde{B}(x,\eta) = \widetilde{b}(x) + \sum_{y\in V}\widetilde{a}(x,y)\eta(y) - m(x)\eta(x)^\lambda$$

where $\widetilde{b} : V \longrightarrow \mathbb{R}_+$ and $\widetilde{a} : V\times V \longrightarrow \mathbb{R}$ are given by

$$\widetilde{b}(x) = b(x) + \sum_{y\in V}\psi(x,y)\int_{\mathcal{X}\setminus\{0\}}\nu(x)\nu(y)H_2(d\nu),$$

$$\widetilde{a}(x,y) = a(x,y) + \mathbb{1}_{\{x\neq y\}} g(y) \int_{\mathcal{X}\backslash\{0\}} \nu(x) H_1(y,d\nu) + \varphi(x,y) \int_{\mathcal{X}\backslash\{0\}} \nu(x) H_2(d\nu).$$

Hence (5) is satisfied, provided that $\lambda = 1$ and $\inf_{x\in V} m(x) > C_1$ where C_1 is the constant from (13). Using the previous remark, we may verify such a condition for particular classes of weight functions v and kernels a.

Formulation (2) contains classical multi-type continuous-state branching processes with immigration as a particular case (when V is finite). The following example shows that it also includes their infinite-dimensional analogues as studied in [9, 19] in terms of Laplace transforms. In such a case, we assume that the immigration structure is given by the state-independent immigration mechanism $\rho(x,\eta,\nu) = \widetilde{\rho}(x,\nu)$, where $x \in V$.

Example (infinite-type continuous-state branching process with immigration). Assume conditions (i) – (iii) of Proposition 2.1 with $m(x) = 0$, and let $\widetilde{\rho} : V \times \mathcal{X}\backslash\{0\} \longrightarrow \mathbb{R}_+$ be measurable such that

$$\int_{\mathcal{X}\backslash\{0\}} \sum_{x\in V} v(x)\nu(x)\widetilde{\rho}(x,\nu) H_2(d\nu) < \infty$$

where $H_2(d\nu)$ is a σ-finite measure on $\mathcal{X}\backslash\{0\}$. Then conditions (A1) – (A6) are satisfied for $\rho(x,\eta,\nu) = \widetilde{\rho}(x,\nu)$. The corresponding process is an infinite-type continuous-state branching process with immigration where V denotes the countable set of different types of the population.

(a) If there exists, in addition, a constant $A > 0$ such that

$$\sum_{x\in V} \widetilde{a}(x,y) v(x) \leq -A v(y), \qquad y \in V,$$

then Theorem 1.2 is applicable, and the process converges to the unique invariant distribution.

(b) Suppose that $V = \mathbb{Z}^d$ is equipped with the 1-norm $|\cdot|_1$. If $b(x) = 0$, $H_2(d\nu) = 0$, v satisfies (9) with $\mathrm{dist}(x_0, x) = |x|_1$, and there exists $R > 0$ such that $a(x,y) = 0$ holds for $|x-y|_1 > R$, then Theorem 1.3 is applicable and the process has at most linear growth.

Below, we extend this setting to processes with interactions. For the sake of simplicity, we restrict our attention towards cylindrical branching and immigration measures H_1, H_2, which constitutes a natural assumption when V contains infinitely many sites.

Remark 2.3. Suppose that the family of measures $(H_1(x,d\nu))_{x\in V}$ and $H_2(d\nu)$ on $\mathcal{X}\backslash\{0\}$ are given by

$$H_1(x,d\nu) = \int_{(0,\infty)} \delta_{z\delta_x}(d\nu)\mu_{x,x}(dz) + \sum_{y\in V\backslash\{x\}} \int_{(0,\infty)} \delta_{z\delta_y}(d\nu)\mu_{x,y}(dz),$$

$$H_2(d\nu) = \sum_{x\in V} \int_{(0,\infty)} \delta_{z\delta_x}(d\nu)\sigma_x(dz)$$

where $(\mu_{x,y})_{x,y\in V}$ and $(\sigma_x)_{x\in V}$ are Lévy measures on $(0,\infty)$ satisfying

$$\sum_{x\in V} v(x) g(x) \int_{(0,1]} z^2 \mu_{x,x}(dz) < \infty,$$

$$\sup_{x\in V} g(x) \int_{(1,\infty)} z\mu_{x,x}(dz) + \sup_{x\in V} \frac{g(x)}{v(x)} \sum_{y\in V\backslash\{x\}} v(y) \int_{(0,\infty)} z\mu_{x,y}(dz) < \infty.$$

Then condition (iii) of Proposition 2.1 is satisfied. Moreover, if

$$\sup_{y\in V} \frac{1}{v(y)} \sum_{x\in V} v(x)\varphi(x,y) \int_{(0,\infty)} z\sigma_x(dz) < \infty$$

$$\sum_{x\in V} v(x) \int_{(0,\infty)} z\sigma_x(dz) + \sum_{x\in V} v(x)\psi(x,x) \int_{(0,\infty)} z^2\sigma_x(dz) < \infty$$

hold, then also condition (iv) of Proposition 2.1 is satisfied.

Proof. Let us remark that $\int_{\{\|\nu\|\leq 1\}\backslash\{0\}} \nu(x)^2 H_1(x,d\nu) = \int_{(0,1]} z^2 \mu_{x,x}(dz) < \infty$ and $\int_{\{\|\nu\|>1\}} \nu(x) H_1(x,d\nu) = \int_{(1,\infty)} z\mu_{x,x}(dz) < \infty$. Moreover, it is easy to see that

$$\int_{\mathcal{X}\backslash\{0\}} \sum_{y\in V\backslash\{x\}} v(y)\nu(y) H_1(x,d\nu) = \sum_{y\in V\backslash\{x\}} v(y) \int_{(0,\infty)} z\mu_{x,y}(dz).$$

This shows that condition (iii) of Proposition 2.1 is satisfied. Condition (iv) therein follows from

$$\int_{\mathcal{X}\backslash\{0\}} \sum_{x\in V} v(x)\nu(x)\varphi(x,y) H_2(d\nu) \leq \sum_{w\in V} v(w)\varphi(w,y) \int_{\mathcal{X}\backslash\{0\}} z\sigma_w(dz) \leq C_\rho v(y)$$

for some constant $C_\rho > 0$, and similarly

$$\sum_{y\in V} \int_{\mathcal{X}\backslash\{0\}} \sum_{x\in V} v(x)\nu(x)\psi(x,y)\nu(y) H_2(d\nu) = \sum_{w\in V} \int_{(0,\infty)} v(w)\psi(w,w) \int_{(0,\infty)} z^2 \sigma_w(dz) < \infty.$$

□

Our next example extends the infinite-type continuous-state branching process from Example 2 towards local interactions in the drift. It extends, in particular, [10] to the infinite-dimensional case.

Example (Local branching process with local competition). Let $V = \mathbb{Z}^d$, and consider the following process on $\mathcal{X}$ given by the strong solution of

$$d\eta_t(x) = \left(\sum_{y\in V} a(x,y)\eta_t(y) - m(x)\eta(x)^\lambda \right) dt + \sqrt{c(x)\eta_t(x)} dB_t(x) + (g(x)\eta_t(x))^{\frac{1}{\alpha(x)}} dZ_t(x) + dJ_t(x)$$

where $(B_t(x))_{t\geq 0}$ is family of independent one-dimensional Brownian motions, $(J_t(x))_{t\geq 0}$ is a family of independent Lévy subordinators on $\mathbb{R}_+$ with Lévy measures σ_x and drift $b(x) \geq 0$, and $(Z_t(x))_{t\geq 0}$ is a family of independent spectrally positive pure-jump Lévy processes with Lévy measure

$$\mu_{\alpha(x)}(dz) = \mathbb{1}_{(0,\infty)}(z) f(x) \frac{dz}{z^{1+\alpha(x)}}$$

where $\alpha(x) \in (1,2)$ and normalization constant

$$f(x) = \int_0^\infty \left(e^{-z} - 1 + z\right) z^{-1-\alpha(x)} dz = \frac{\Gamma(2-\alpha(x))}{\alpha(x)(\alpha(x)-1)}.$$

By letting H_1, H_2 be given as in previous remark with $\mu_{x,x} = \mu_{\alpha(x)}$, $\mu_{x,z} = 0$, and $\rho(x,\eta,\nu) \equiv 1$, it is easy to see that this model is equivalent in law to (2). Assume $b, \lambda, m \geq 0$, that $a : \mathbb{Z}^d \times \mathbb{Z}^d \longrightarrow \mathbb{R}$ satisfies $a(x,y) \geq 0$ for $x \neq y$, (13) holds, that

$$\sum_{x\in\mathbb{Z}^d} v(x) \left(b(x) + c(x) + \int_{(0,\infty)} z\sigma_x(dz) \right) < \infty, \tag{15}$$

and

$$\sum_{x\in\mathbb{Z}^d} v(x)g(x) \frac{\Gamma(2-\alpha(x))}{\alpha(x)(\alpha(x)-1)(2-\alpha(x))} < \infty, \qquad \sup_{x\in\mathbb{Z}^d} g(x) \frac{\Gamma(2-\alpha(x))}{\alpha(x)(\alpha(x)-1)^2} < \infty. \tag{16}$$

Then conditions (A1) – (A6) are satisfied.

(a) If $\lambda = 1$ and there exists $A > 0$ such that

$$\sum_{x\in V} a(x,y)v(x) \leq -Av(y) + m(y)v(y)$$

then Theorem 1.2 is applicable, and the process converges to its unique limit distribution.

(b) If $b(x) \equiv 0$, $\sigma_x \equiv 0$, v satisfies (9) with $\mathrm{dist}(x_0, x) = |x|_1$, and there exists $R > 0$ such that $a(x, y) = 0$ for $|x - y|_1 > R$, then Theorem 1.3 is applicable.

Proof. Let us show it is a particular case of Proposition 2.1. Conditions (i) and (ii) therein are evident. Condition (iii) follows from previous remark combined with $\int_{(0,1]} z^2 \mu_{\alpha(x)}(dz) = \frac{f(x)}{2-\alpha(x)}$ and $\int_{(1,\infty)} z \mu_{\alpha(x)}(dz) = \frac{f(x)}{\alpha(x)-1}$. Hence, conditions (A1) – (A4) are satisfied. Condition (A5) is trivial since $\rho \equiv 1$ while (A6) follows from

$$\int_{\mathcal{X}\setminus\{0\}} \sum_{x \in V} v(x)\nu(x) H_2(d\nu) = \sum_{w \in V} v(w) \int_{(0,\infty)} z\sigma_w(dz) < \infty.$$

Assertions (a) and (b) are left for the reader. □

Assumptions (15) and (16) are natural to guarantee that $(J_t(x))_{t\geq 0}$ and the stochastic integrals against $(B_t(x))_{t\geq 0}$ and $(Z_t(x))_{t\geq 0}$ take values in $\mathcal{X}$. For constant g, (16) implies that $\alpha(x)$ is bounded away from 1. If $v \equiv 1$, then (16) also implies that $\alpha(x)$ needs to be bounded away from 2. However, in general, α may approach 1 and 2 provided that the singularities in the denominator are compensated by $g(x)$ and $v(x)$.

Let us close this presentation with two additional discrete examples previously studied in the literature.

Example (nearest-neighbor continuous-state branching process with unit jumps). Consider $V = \mathbb{Z}^d$, $c(x,t) \equiv c_0 t$, $g(x,t) \equiv g_0 t$, and $\rho \equiv 0$, where $c_0, g_0 \geq 0$. Let

$$B(x,\eta) = \sum_{y \in \mathbb{Z}^d : |y-x|=1} \eta(y),$$

$H_1(x,\cdot) = \sum_{y\in\mathbb{Z}^d:|y-x|=1} \delta_{\delta_y}$, and $v(x) = e^{-|x|_1}$, where $|x|_1$ is the ℓ^1 norm of $x \in \mathbb{Z}^d$. This gives a nearest-neighbor continuous-space branching process with unit jumps. It is straightforward to check that (A1)–(A6) are satisfied and Theorems 1.1 and 1.3 hold.

Example (branching random walk). Take $V = \mathbb{Z}^d$ and $v(x) = e^{-|x|_1}$. Assuming that for every $x \in V$ the measure $H_1(x,\cdot)$ is concentrated on integer-valued elements of $\mathcal{X}$ and taking $c \equiv 0$, $\rho \equiv 0$, $g(x,s) \equiv s$, and

$$B(x,\eta) = \eta(x) \int_{\{\|\nu\|\geq 1\}} \nu(x) H_1(x, d\nu).$$

Suppose further that H_1 satisfies

$$\sup_{x\in\mathbb{Z}^d} \int_{\{\|\nu\|\geq 1\}} \nu(x) H_1(x,d\nu) + \sup_{x\in\mathbb{Z}^d} e^{-|x|_1} \int_{\mathcal{X}\setminus\{0\}} \sum_{y \in V\setminus\{x\}} e^{-|y|_1} \nu(y) H_1(x,d\nu) =: C < \infty.$$

Then conditions (A1) – (A6) are satisfied with $C_1 = C_4 = C_6 = C$, and $C_2 = C_3 = C_5 = 0$, and we obtain a continuous-time discrete-space branching random walk (see e.g. [3, 5]). The process is inhomogeneous in space and can fit in the framework of a branching random walk in a random environment (see, e.g. [29]) if the measures $H_1(x,\cdot)$ are additionally randomized. Finally, if

$$\int_{\mathcal{X}\setminus\{0\}} \nu(x) H_1(y, d\nu) = 0, \qquad |x-y|_1 > R$$

holds for some $R > 0$, then Theorem 1.3 is applicable.

3. Non-explosion and first-moment estimate

A solution η of (2) with lifetime ζ consists of a stopping time ζ and a process $(\eta_t)_{t\in[0,\zeta)}$ such that (2) is satisfied on $\{t \leq \tau_m(\eta)\}$ for each $m \geq 1$, where

$$\tau_m(\eta) = \inf\{t \in [0,\zeta) \mid \|\eta_t\| > m\}. \tag{17}$$

with the convention $\inf \emptyset = \infty$. Clearly, $\tau_m(\eta)$ is an increasing sequence of stopping times. Let us first prove that each solution of (2) is always conservative.

Theorem 3.1. *Suppose that (A1) – (A4), and (A6) are satisfied. Let* $(\eta_t)_{t\in[0,\zeta)}$ *be a solution of* (2) *with lifetime* ζ. *Let* $\tau_m = \tau_m(\eta)$ *be the stopping time defined in* (17). *Then* $\tau_m \nearrow \infty$ *a.s. as* $m \to \infty$.

Proof. Using Lemma A.1, we can assume without loss of generality that $v \equiv 1$. The definition of τ_m implies that $\tau_m \leq \tau_{m+1}$ holds for each $m \geq 1$. Define $\sup_{m\geq 1}\tau_m = \tau$. Then we have to prove that $\mathbb{P}[\tau = \infty] = 1$. Fix $T > 0$, then $\mathbb{P}[\tau \leq T] = \lim_{m\to\infty}\mathbb{P}[\tau_m \leq T]$ and hence it suffices to prove that $\mathbb{P}[\tau_m \leq T] \to 0$ as $m \to \infty$. For this purpose, we note that

$$\mathbb{P}[\tau_m \leq T] = \mathbb{P}\left[\sup_{t\in[0,T]}\|\eta_t\| > m\right] \leq \frac{1}{m}\mathbb{E}\left[\sup_{t\in[0,T]}\|\eta_t\|\right]\frac{1}{m}\sum_{x\in V}\mathbb{E}\left[\sup_{t\in[0,T]}\eta_t(x)\right]. \tag{18}$$

Let $x \in V$ and let $\mathcal{M}_t(x)$ be the local martingale defined by

$$\mathcal{M}_t(x) = \int_0^t \sqrt{2c(x,\eta_s(x))}dW_s(x) + \int_0^t\int_{\mathcal{X}\backslash\{0\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\leq g(x,\eta_{s-}(x))\}}\widetilde{N}_x(ds,d\nu,du).$$

Using the Burkholder-Davis-Gundy inequality combined with (A2) and (A3), we find that

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_m]}|\mathcal{M}_t(x)|\right] &\leq \left(\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_m]}\left|\int_0^t\sqrt{2c(x,\eta_s(x))}dW_s(x)\right|^2\right]\right)^{1/2} \\
&\quad + \left(\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_m]}\left|\int_0^t\int_{\{\|\nu\|\leq 1\}\backslash\{0\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\leq g(x,\eta_{s-}(x))\}}\widetilde{N}_x(ds,d\nu,du)\right|^2\right]\right)^{1/2} \\
&\quad + \mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_m]}\left|\int_0^t\int_{\{\|\nu\|>1\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\leq g(x,\eta_{s-}(x))\}}\widetilde{N}_x(ds,d\nu,du)\right|\right] \\
&\leq \sqrt{8}\left(\mathbb{E}\left[\int_0^T\mathbb{1}_{[0,\tau_m]}(s)c(x,\eta_s(x))ds\right]\right)^{1/2} \\
&\quad + 2\left(\mathbb{E}\left[\int_0^T\int_{\{\|\nu\|\leq 1\}\backslash\{0\}}\int_{\mathbb{R}_+}\mathbb{1}_{[0,\tau_m]}(s)\nu(x)^2\mathbb{1}_{\{u\leq g(x,\eta_{s-}(x))\}}\widehat{N}_x(ds,d\nu,du)\right]\right)^{1/2} \\
&\quad + 2\mathbb{E}\left[\int_0^T\int_{\{\|\nu\|>1\}}\int_{\mathbb{R}_+}\mathbb{1}_{[0,\tau_m]}(s)\nu(x)\mathbb{1}_{\{u\leq g(x,\eta_{s-}(x))\}}\widehat{N}_x(ds,d\nu,du)\right] \\
&\leq \sqrt{8}\sqrt{C_2(x)}\left(\int_0^T\mathbb{E}\left[\sup_{r\in[0,s\wedge\tau_m]}\eta_r(x)\right]ds\right)^{1/2} \\
&\quad + 2\sqrt{C_3(x)}\left(\int_{\{\|\nu\|\leq 1\}\backslash\{0\}}\nu(x)^2H_1(x,d\nu)\right)^{1/2}\left(\int_0^T\mathbb{E}\left[\sup_{r\in[0,s\wedge\tau_m]}\eta_r(x)\right]ds\right)^{1/2} \\
&\quad + 2C_3(x)\left(\int_{\{\|\nu\|>1\}}\nu(x)H_1(x,d\nu)\right)\int_0^T\mathbb{E}\left[\sup_{r\in[0,s\wedge\tau_m]}\eta_r(x)\right]ds
\end{aligned}$$

Define $f_m(t;x) = \mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_m]}\eta_s(x)\right]$. Summing over $x \in V$ we obtain

$$\begin{aligned}
\sum_{x\in V}\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_m]}|\mathcal{M}_t(x)|\right] &\leq \sqrt{8}\sum_{x\in V}\sqrt{C_2(x)}\left(\int_0^T f_m(s;x)ds\right)^{1/2} \\
&+ 2\sum_{x\in V}\sqrt{C_3(x)}\left(\int_{\{\|\nu\|\leq 1\}\backslash\{0\}}\nu(x)^2H_1(x,d\nu)\right)^{1/2}\left(\int_0^T f_m(s;x)ds\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{x\in V} C_3(x)\left(\int_{\{\|\nu\|>1\}}\nu(x)H_1(x,d\nu)\right)\int_0^T f_m(s;x)ds\\
&\le \sqrt{8}\left(\sum_{y\in V} C_2(y)\right)\int_0^T\sum_{x\in V} f_m(s;x)ds\\
&+2\left(\sum_{y\in V} C_3(y)\int_{\{\|\nu\|\le 1\}\setminus\{0\}}\nu(y)^2 H_1(y,d\nu)\right)\int_0^T\sum_{x\in V} f_m(s;x)ds+2C_4\int_0^T\sum_{x\in V} f_m(s;x)ds\\
&=: c_0\int_0^T\sum_{x\in V} f_m(s;x)ds
\end{aligned}
$$

with a constant $c_0\in(0,\infty)$. Using (2) combined with (A6), we find that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_m]}\eta_t(x)\right] &\le \mathbb{E}[\eta_0(x)]+\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_m]}|\mathcal{M}_t(x)|\right]+\mathbb{E}\left[\int_0^T \mathbb{1}_{[0,\tau_m]}(s)|B_0(x,\eta_s)|ds\right]\\
&+\sum_{y\in V\setminus\{x\}}\mathbb{E}\left[\int_0^T\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\mathbb{1}_{[0,\tau_m]}(s)\nu(x)\mathbb{1}_{\{u\le g(y,\eta_{s-}(y))\}}N_y(ds,d\nu,du)\right]\\
&+\mathbb{E}\left[\int_0^T\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\mathbb{1}_{[0,\tau_m]}(s)\nu(x)\mathbb{1}_{\{u\le\rho(x,\eta_{s-},\nu)\}}M(ds,d\nu,du)\right]\\
&\le \mathbb{E}[\eta_0(x)]+\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_m]}|\mathcal{M}_t(x)|\right]+\int_0^T\mathbb{E}\left[\mathbb{1}_{[0,\tau_m]}(s)|B_0(x,\eta_s)|\right]ds\\
&+\sum_{y\in V\setminus\{x\}} C_3(y)\left(\int_{\mathcal{X}\setminus\{0\}}\nu(x)H_1(y,d\nu)\right)\int_0^T f_m(s;y)ds\\
&+\int_0^T\int_{\mathcal{X}\setminus\{0\}}\nu(x)\mathbb{E}\left[\mathbb{1}_{[0,\tau_m]}(s)\rho(x,\eta_s,\nu)\right]H_2(d\nu)ds.
\end{aligned}
$$

After summation over x, let us estimate the last three terms separately. Recall that $\|B_0(\cdot,\eta)\|=\sum_{x\in V}B_0(x,\eta)$. For the first one, we obtain by (A6)

$$
\sum_{x\in V}\int_0^T\mathbb{E}\left[\mathbb{1}_{[0,\tau_m]}(s)|B_0(x,\eta_s)|\right]ds=\int_0^T\mathbb{E}\left[\mathbb{1}_{[0,\tau_m]}(s)\|B_0(\cdot,\eta_s)\|\right]ds\le C_6T+C_6\int_0^T\sum_{x\in V}f_m(s;x)ds.
$$

The second term given by (A4)

$$
\begin{aligned}
&\sum_{x\in V}\sum_{y\in V\setminus\{x\}} C_3(y)\left(\int_{\mathcal{X}\setminus\{0\}}\nu(x)H_1(y,d\nu)\right)\int_0^T f_m(s;y)ds\\
&=\sum_{y\in V} C_3(y)\left(\int_{\mathcal{X}\setminus\{0\}}\sum_{x\in V\setminus\{y\}}\nu(x)H_1(y,d\nu)\right)\int_0^T f_m(s;y)ds\le C_4\int_0^T\sum_{y\in V} f_m(s;y)ds.
\end{aligned}
$$

Finally, for the last term, we obtain by (A6)

$$
\begin{aligned}
&\sum_{x\in V}\int_0^T\int_{\mathcal{X}\setminus\{0\}}\nu(x)\mathbb{E}\left[\mathbb{1}_{[0,\tau_m]}(s)\rho(x,\eta_s,\nu)\right]H_2(d\nu)ds\\
&=\int_0^T\mathbb{E}\left[\mathbb{1}_{[0,\tau_m]}(s)\int_{\mathcal{X}\setminus\{0\}}\sum_{x\in V}\nu(x)\rho(x,\eta_s,\nu)H_2(d\nu)\right]ds
\end{aligned}
$$

$$\le C_6 T + C_6 \int_0^T \sum_{x\in V} f_m(s;x)ds.$$

Together, these inequalities yield

$$\sum_{x\in V} f_m(T;x) \le \mathbb{E}[\|\eta_0\|] + 2C_6 T + (c_0 + 2C_6 + C_4)\int_0^T \sum_{x\in V} f_m(s;x)ds.$$

The Gronwall inequality yields

$$\sum_{x\in V} v(x) f_m(T;x) \le \left(\mathbb{E}[\|\eta_0\|] + 2C_6 T\right) e^{(c_0+2C_6+C_4)T}.$$

Letting $m \to \infty$ and using Fatou's lemma gives

$$\sum_{x\in V} \mathbb{E}\left[\sup_{t\in[0,T]} \eta_t(x)\right] \le \sup_{m\ge 1} \sum_{x\in V} f_m(T;x) \le \left(\mathbb{E}[\|\eta_0\|] + 2C_6 T\right) e^{(c_0+2C_6+C_4)T} < \infty.$$

With this, the theorem statement follows from (18). □

Next, we prove a simple but useful observation used for the localization of coefficients.

Lemma 3.2. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ be a stochastic basis with the usual conditions and let $(\eta_t)_{t\ge 0}$ be an $(\mathcal{F}_t)_{t\ge 0}$-adapted cadlag process such that $\|\eta_t\| < \infty$ holds a.s. Define $\tau_m(\eta)$ as in (17) with $\zeta = +\infty$. Then $(\tau_m(\eta))_{m\in\mathbb{N}}$ is an increasing sequence of stopping times satisfying $\tau_m(\eta) \nearrow \infty$ a.s. as $m \to \infty$. Finally, one has $\|\eta_{t-}\| \le m$ and $\eta_{t-}(x) \le \frac{m}{v(x)}$ a.s. for each $t \in [0, \tau_m]$ and $x \in V$.*

Proof. Since $(\mathcal{F}_t)_{t\ge 0}$ is right-continuous and $(\eta_t)_{t\ge 0}$ has cadlag paths, it follows that $\tau_m(\eta)$ is an $(\mathcal{F}_t)_{t\ge 0}$-stopping time. Next observe that, by definition, $\tau_m(\eta) \le \tau_{m+1}(\eta)$ holds a.s. for each $m \in \mathbb{N}$. Let $\tau(\eta) := \sup_{m\in\mathbb{N}} \tau_m(\eta)$. Then for all $t > 0$ we obtain

$$\mathbb{P}[\tau(\eta) > t] = \lim_{m\to\infty} \mathbb{P}[\tau_m(\eta) > t] = \lim_{m\to\infty} \mathbb{P}[\|\eta_t\| \le m] = \mathbb{P}[\|\eta_t\| < \infty] = 1.$$

Letting $t \to \infty$ yields $\tau(\eta) = \infty$ a.s.. The property $\|\eta_{t-}\| \le m$ for $t \in [0, \tau_m]$ holds by definition of τ_m, while the second inequality follows from $\eta_{t-}(x) \le \frac{1}{v(x)}\|\eta_{t-}\| \le \frac{m}{v(x)}$. □

Finally, we prove a first-moment estimate for the solutions of (2) with parameters depending locally uniformly on the constants appearing in (A1) – (A6).

Theorem 3.3. *Suppose that (A1) – (A4), and (A6) are satisfied. Then there exists a constant $C > 0$ such that each weak solution $(\eta_t)_{t\ge 0}$ of (2) satisfies*

$$\mathbb{E}[\|\eta_t\|] \le (1 + \mathbb{E}[\|\eta_0\|])\, e^{Ct}, \qquad t \ge 0. \tag{19}$$

Furthermore, let $(\xi_t)_{t\ge 0}$ be a weak solution to (2) with different functions $\tilde{B}, \tilde{B}_0, \tilde{B}_1, \tilde{c}, \tilde{g}, \tilde{\rho}$ instead of B, B_0, B_1, c, g, ρ, respectively. Assume that (A1) – (A4) and (A6) are satisfied for $\tilde{B}, \tilde{B}_0, \tilde{B}_1, \tilde{c}, \tilde{g}, \tilde{\rho}$, and for all $\alpha, \beta \in \mathcal{X}$, $\alpha \le \beta$, $x \in V$, and $0 \le s \le t$ we have $\tilde{B}_0(x,\alpha) \le B_0(x,\beta)$, $\tilde{c}(x,s) \le c(x,t)$, $\tilde{g}(x,s) \le g(x,t)$, $\tilde{\rho}(x,\alpha,\nu) \le \rho(x,\beta,\nu)$. Then

$$\mathbb{E}[\|\xi_t\|] \le (1 + \mathbb{E}[\|\xi_0\|])\, e^{Ct}, \qquad t \ge 0. \tag{20}$$

with the same constant C as in (19).

Proof. Again, using Lemma A.1, we suppose without loss of generality that $v \equiv 1$. The main part of the proof is establishing (19). Let $\tau_m(\eta)$ be the stopping time defined in Lemma 3.2. Observe that

$$\eta_t(x) = \eta_0(x) + \int_0^t \left(B(x,\eta_s) + \sum_{y\in V\setminus\{x\}} \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x) \mathbb{1}_{\{u \le g(y,\eta_{s-}(y))\}} H_1(y, d\nu) du \right) ds$$

$$+ \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x) \mathbb{1}_{\{u \le \rho(x,\eta_{s-},\nu)\}} M(ds, d\nu, du) + \mathcal{M}_t(x)$$

with $(\mathcal{M}_t(x))_{t\geq 0}$ given by

$$\begin{aligned}\mathcal{M}_t(x) := &\int_0^t \sqrt{2c(x,\eta_s(x))}dW_s(x)\\ &+\int_0^t\int_{\{\|\nu\|\leq 1\}\setminus\{0\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\leq g(x,\eta_{s-}(x))\}}\widetilde{N}_x(ds,d\nu,du)\\ &+\int_0^t\int_{\{\|\nu\|>1\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\leq g(x,\eta_{s-}(x))\}}\widetilde{N}_x(ds,d\nu,du)\\ &+\sum_{y\in V\setminus\{x\}}\int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\leq g(y,\eta_{s-}(y))\}}\widetilde{N}_y(ds,d\nu,du).\end{aligned}$$

Next we prove that $(\mathcal{M}_{t\wedge\tau_m}(x))_{t\geq 0}$ is a martingale for each $m\geq 1$ and $x\in V$. Indeed, the first two terms are square integrable martingales since by (A2) we have

$$\mathbb{E}\left[\left|\int_0^{t\wedge\tau_m}\sqrt{2c(x,\eta_s(x))}dW_s(x)\right|^2\right]=\mathbb{E}\left[\int_0^{t\wedge\tau_m}2c(x,\eta_s(x))ds\right]\leq 2C_2(x)mt<\infty, \tag{21}$$

and by (A3) and (A4) also

$$\begin{aligned}&\mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\{\|\nu\|\leq 1\}\setminus\{0\}}\int_{\mathbb{R}_+}\left|\nu(x)\mathbb{1}_{\{u\leq g(x,\eta_{s-}(x))\}}\right|^2 duH_1(x,d\nu)ds\right]\\ &=\mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\{\|\nu\|\leq 1\}\setminus\{0\}}\nu(x)^2 g(x,\eta_{s-}(x))H_1(x,d\nu)ds\right]\leq C_3(x)t\int_{\{\|\nu\|\leq 1\}\setminus\{0\}}\nu(x)^2H_1(x,d\nu)<\infty.\end{aligned}$$

The third term is a martingale due to (A4) and

$$\mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\{\|\nu\|>1\}}\int_{\mathbb{R}_+}\left|\nu(x)\mathbb{1}_{\{u\leq g(x,\eta_{s-}(x))\}}\right| duH_1(x,d\nu)ds\right]\leq C_3(x)t\int_{\{\|\nu\|>1\}}\nu(x)H_1(x,d\nu)<\infty.$$

Finally, the last term is a martingale since by (A3)

$$\begin{aligned}&\sum_{y\in V\setminus\{x\}}\mathbb{E}\left[\left|\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\leq g(y,\eta_{s-}(y))\}}\widetilde{N}_y(ds,d\nu,du)\right|\right]\\ &\leq 2\sum_{y\in V\setminus\{x\}}\mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}}\nu(x)g(y,\eta_{s-}(y))H_1(y,d\nu)ds\right]\\ &\leq 2\sum_{y\in V\setminus\{x\}}C_3(y)\mathbb{E}\left[\int_0^{t\wedge\tau_m}\eta_s(y)ds\right]\int_{\mathcal{X}\setminus\{0\}}\nu(x)v(x)H_1(y,d\nu)\\ &\leq 2\sum_{y\in V\setminus\{x\}}C_3(y)\mathbb{E}\left[\int_0^{t\wedge\tau_m}\eta_s(y)ds\right]\int_{\mathcal{X}\setminus\{0\}}\sum_{w\in V\setminus\{y\}}\nu(w)v(w)H_1(y,d\nu)\\ &\leq C_4\sum_{y\in V\setminus\{x\}}\mathbb{E}\left[\int_0^{t\wedge\tau_m}\eta_s(y)ds\right]v(y)\\ &\leq C_4\mathbb{E}\left[\int_0^{t\wedge\tau_m}\|\eta_s\|ds\right]\leq C_4mt<\infty.\end{aligned}$$

This proves that $(\mathcal{M}_{t\wedge\tau_m}(x))_{t\geq 0}$ is a martingale. Hence, taking expectations and using optimal stopping for integrable martingales gives

$$\begin{aligned}
\mathbb{E}[\eta_{t\wedge\tau_m}(x)] &= \mathbb{E}[\eta_0(x)] + \mathbb{E}\left[\int_0^{t\wedge\tau_m} (B_0(x,\eta_s) - B_1(x,\eta_s(x)))ds\right] + \mathbb{E}\left[\sum_{y\in V\setminus\{x\}} \int_{\mathcal{X}\setminus\{0\})} \nu(x)g(y,\eta_{s-}(y))H_1(y,d\nu)\right] \\
&\quad + \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}} \nu(x)\rho(x,\eta_{s-},\nu)H_2(d\nu)ds\right] \\
&\leq \mathbb{E}[\eta_0(x)] + \mathbb{E}\left[\int_0^{t\wedge\tau_m} B_0(x,\eta_s)ds\right] + \mathbb{E}\left[\sum_{y\in V\setminus\{x\}} \int_{\mathcal{X}\setminus\{0\}} \nu(x)g(y,\eta_{s-}(y))H_1(y,d\nu)\right] \\
&\quad + \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}} \nu(x)\rho(x,\eta_{s-},\nu)H_2(d\nu)ds\right]
\end{aligned}$$

where we have used that B_1 is non-decreasing so that $B_1(x,\eta(x)) \geq B_1(x,0) = 0$. This yields by (A3), (A4), and (A6)

$$\begin{aligned}
\mathbb{E}\left[\|\eta_{t\wedge\tau_m}\|\right] &= \sum_{x\in V} \mathbb{E}[\eta_{t\wedge\tau_m}(x)] \\
&\leq \mathbb{E}[\|\eta_0\|] + \mathbb{E}\left[\int_0^{t\wedge\tau_m} \sum_{x\in V} B_0(x,\eta_s)ds\right] + \mathbb{E}\left[\sum_{x\in V}\sum_{y\in V\setminus\{x\}} \int_{\mathcal{X}\setminus\{0\}} \nu(x)g(y,\eta_{s-}(y))H_1(y,d\nu)\right] \\
&\quad + \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}} \sum_{x\in V} \nu(x)\rho(x,\eta_{s-},\nu)H_2(d\nu)ds\right] \\
&\leq \mathbb{E}[\|\eta_0\|] + 2C_6 t + (2C_6 + C_4)\int_0^t \mathbb{E}[\|\eta_{s\wedge\tau_m}\|]ds
\end{aligned}$$

where we have used (A3) and (A4) so that

$$\begin{aligned}
\sum_{x\in V}\sum_{y\in V\setminus\{x\}} \int_{\mathcal{X}\setminus\{0\}} \nu(x)g(y,\eta_{s-}(y))H_1(y,d\nu) &= \sum_{y\in V} g(y,\eta_{s-}(y)) \int_{\mathcal{X}\setminus\{0\}} \sum_{x\in V\setminus\{y\}} \nu(x)H_1(y,d\nu) \\
&\leq \sum_{y\in V} \frac{C_4}{C_3(y)} g(y,\eta_{s-}(y)) \leq C_4\|\eta_{s-}\|.
\end{aligned}$$

Inequality (19) now follows from the Gronwall lemma. The proof of (20) follows the same path; we just need to replace $\tilde{B}, \tilde{B}_0, \tilde{B}_1, \tilde{c}, \tilde{g}, \tilde{\rho}$ with B, B_0, B_1, c, g, ρ respectively along the way. For example, instead of (21) we write

$$\begin{aligned}
\mathbb{E}\left[\left|\int_0^{t\wedge\tau_m} \sqrt{2\tilde{c}(x,\xi_s(x))}dW_s(x)\right|^2\right] &= \mathbb{E}\left[\int_0^{t\wedge\tau_m} 2\tilde{c}(x,\xi_s(x))ds\right] \\
&\leq \mathbb{E}\left[\int_0^{t\wedge\tau_m} 2c(x,\xi_s(x))ds\right] \leq 2C_2(x)mt < \infty.
\end{aligned}$$

□

4. Pathwise uniqueness

In this section, we prove the pathwise uniqueness of the solution under slightly weaker conditions, i.e., we consider:

(A1') The drift coefficient $B(x,\eta)$ has the form $B(x,\eta) = B_0(x,\eta) - B_1(x,\eta(x))$ where $B_0(x,\cdot) : \mathcal{X} \longrightarrow \mathbb{R}_+$ and $B_1(x,\cdot) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are measurable mappings for each $x \in V$. Moreover, for each $R > 0$ there exists a constant $C_1(R) > 0$ such that

$$\|B_0(\cdot,\eta) - B_0(\cdot,\xi)\| \leq C_1(R)\|\eta - \xi\|, \qquad x \in V,$$

holds for all $\eta, \xi \in \mathcal{X}$ with $\|\eta\|, \|\xi\| \leq R$. Finally, the function $\mathbb{R}_+ \ni t \longmapsto B_1(x,t)$ is continuous and non-decreasing satisfying $B_1(x,0) = 0$ for each $x \in V$.

(A5') For each $R > 0$ there exists a constant $C_5(R) > 0$ such that

$$\int_{\mathcal{X}\setminus\{0\}} \sum_{x\in V} v(x)\nu(x)|\rho(x,\eta,\nu) - \rho(x,\xi,\nu)| H_2(d\nu) \leq C_5(R)\|\eta - \xi\|$$

holds for all $\eta, \xi \in \mathcal{X}$ with $\|\eta\|, \|\xi\| \leq R$.

In contrast to (A1) and (A5), the above conditions do not require that B and ρ are monotone with respect to the configuration η. The following is our main result on the uniqueness of (2).

Theorem 4.1. *Let $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ be two weak solutions to (2) defined on the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and suppose that conditions (A1'), (A2) – (A4), and (A5') are satisfied. Let $\tau_m(\eta), \tau_m(\xi)$ be the stopping times defined in Lemma 3.2 and set $\tau_m := \tau_m(\eta) \wedge \tau_m(\xi)$. Then*

$$\mathbb{E}[\|\eta_{t\wedge\tau_m} - \xi_{t\wedge\tau_m}\|] \leq \mathbb{E}[\|\eta_0 - \xi_0\|]e^{(C_1(m)+2C_4+C_5(m))t}, \qquad t \geq 0$$

holds for each $m \geq 1$. In particular, if $\eta_0 = \xi_0$ holds a.s., then $\mathbb{P}[\eta_t = \xi_t,\ t \geq 0] = 1$, i.e. pathwise uniqueness among weak solutions to (2) holds.

Proof. By Lemma A.1, we suppose without loss of generality that $v \equiv 1$. Define $\zeta_t := \eta_t - \xi_t$ and fix $x \in V$. Then

$$\begin{aligned}
\zeta_t(x) &= \zeta_0(x) + \int_0^t (B(x,\eta_s) - B(x,\xi_s))\, ds \\
&+ \int_0^t \left(\sqrt{2c(x,\eta_s(x))} - \sqrt{2c(x,\xi_s(x))}\right) dW_s(x) \\
&+ \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x)\left(\mathbb{1}_{\{u\leq g(x,\eta_{s-}(x))\}} - \mathbb{1}_{\{u\leq g(x,\xi_{s-}(x))\}}\right) \widetilde{N}_x(ds,d\nu,du) \\
&+ \sum_{y\in V\setminus\{x\}} \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x)\left(\mathbb{1}_{\{u\leq g(y,\eta_{s-}(y))\}} - \mathbb{1}_{\{u\leq g(y,\xi_{s-}(y))\}}\right) N_y(ds,d\nu,du) \\
&+ \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x)\left(\mathbb{1}_{\{u\leq \rho(x,\eta_{s-},\nu)\}} - \mathbb{1}_{\{u\leq \rho(x,\xi_{s-},\nu)\}}\right) M(ds,d\nu,du).
\end{aligned}$$

Let $\phi_k : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a sequence of twice continuously differentiable functions such that for each $k \geq 1$,

(i) $\phi_k(-z) = \phi_k(z) \nearrow |z|$ as $k \to \infty$,
(ii) $\phi_k'(z) \in [0,1]$ for $z \geq 0$ and $\phi_k'(z) \in [-1,0]$ for $z \leq 0$,
(iii) $\phi_k''(z)|z| \leq 2/k$ holds for all $z \in \mathbb{R}$.

The construction of such a function follows the same arguments as the classical Yamada-Watanabe theorem for pathwise uniqueness. To simplify the notation below, we set $D_h\phi_k(z) := \phi_k(z+h) - \phi_k(z)$ for $z, h \in \mathbb{R}$. Let $z, h \in \mathbb{R}$ such that $zh \geq 0$. Then, using the mean-value theorem, one can check that

(22) $$D_h\phi_k(z) \leq |h|, \quad D_h\phi_k(z) - \phi_k'(z)h \leq \frac{h^2}{k|z|}, \quad \text{and } D_h\phi_k(z) - \phi_k'(z)h \leq |h|.$$

Applying the Itô formula to $\zeta_t(x)$ gives

(23) $$\phi_k(\zeta_t(x)) = \phi_k(\zeta_0(x)) + \sum_{j=1}^{5} \mathcal{R}_j(t) + \mathcal{M}(t),$$

where the processes $\mathcal{R}_1, \dots, \mathcal{R}_5$ are given by

$$\mathcal{R}_1(t) = \int_0^t \phi_k'(\zeta_s(x))\,(B(x,\eta_s) - B(x,\xi_s))\, ds$$

$$\mathcal{R}_2(t) = \frac{1}{2}\int_0^t \phi_k''(\zeta_s(x)) \left(\sqrt{2c(x,\eta_s(x))} - \sqrt{2c(x,\xi_s(x))}\right)^2 ds$$

$$\mathcal{R}_3(t) = \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \left(D_{\Delta_0(x,s)}\phi_k(\zeta_{s-}(x)) - \phi_k'(\zeta_{s-}(x))\Delta_0(x,s)\right) ds H_1(x,d\nu)du$$

$$\mathcal{R}_4(t) = \sum_{y\in V\setminus\{x\}} \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} D_{\Delta_0(y,s)}\phi_k(\zeta_{s-}(x)) ds H_1(y,d\nu)du$$

$$\mathcal{R}_5(t) = \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} D_{\Delta_1(x,s)}\phi_k(\zeta_{s-}(x)) ds H_2(d\nu)du$$

with increments given by

$$\Delta_0(z,s) = \nu(x)\left(\mathbb{1}_{\{u\le g(z,\eta_{s-}(z))\}} - \mathbb{1}_{\{u\le g(z,\xi_{s-}(z))\}}\right),$$
$$\Delta_1(z,s) = \nu(z)\left(\mathbb{1}_{\{u\le \rho(z,\eta_{s-},\nu)\}} - \mathbb{1}_{\{u\le \rho(z,\xi_{s-},\nu)\}}\right).$$

Note that $\Delta_0(z,s)$ and $\Delta_1(z,s)$ also depend on $u\ge 0$. The process $(\mathcal{M}(t))_{t\ge 0}$ given by

$$\begin{aligned}\mathcal{M}(t) &= \int_0^t \phi_k'(\zeta_s(x)) \left(\sqrt{2c(x,\eta_s(x))} - \sqrt{2c(x,\xi_s(x))}\right) dW_s(x) \\ &\quad + \sum_{y\in V} \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} D_{\Delta_0(y,s)}\phi_k(\zeta_{s-}(x)) \widetilde{N}_y(ds,d\nu,du) \\ &\quad + \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} D_{\Delta_1(x,s)}\phi_k(\zeta_{s-}(x)) \widetilde{M}(ds,d\nu,du),\end{aligned} \tag{24}$$

is a local martingale. Here, $\widetilde{M} = M - \widehat{M}$ denotes the compensated Poisson random measure. Note that $\mathcal{R}_j$ and $\mathcal{M}(t)$ also depend on the previously fixed point x. Recall that τ_m satisfies, by Lemma 3.2, $\tau_m \longrightarrow \infty$ and it holds that

$$\eta_{s-}(x), \xi_{s-}(x) \le m \quad \text{and} \quad \|\eta_{s-}\|, \|\xi_{s-}\| \le m, \qquad s\in[0,\tau_m], \;\; x\in V. \tag{25}$$

Using Property (25), it is not difficult to see that $(\mathcal{M}(t\wedge\tau_m))_{t\ge 0}$ is a martingale for each k and each m. For the sake of completeness, the proof is given in the appendix. Below we will show that

$$\begin{aligned}\mathcal{R}_1(t\wedge\tau_m) &\le \int_0^{t\wedge\tau_m} |B_0(x,\eta_s) - B_0(x,\xi_s)| ds, \\ \mathcal{R}_2(t\wedge\tau_m) &\le C_2(x)\frac{2t}{k}, \\ \mathcal{R}_3(t\wedge\tau_m) &\le \frac{C_3(x)t}{k} \int_{\{\|\nu\|\le 1\}\setminus\{0\}} \nu(x)^2 H_1(x,d\nu) \\ &\quad + C_3(x)\left(\int_{\{\|\nu\|>1\}} \nu(x) H_1(x,d\nu)\right) \int_0^{t\wedge\tau_m} |\eta_{s-}(x) - \xi_{s-}(x)| ds,\end{aligned} \tag{26}$$

$$\begin{aligned}\mathcal{R}_4(t\wedge\tau_m) &\le \sum_{y\in V\setminus\{x\}} \left(\int_{\mathcal{X}\setminus\{0\}} \nu(x) H_1(y,d\nu)\right) C_3(y) \int_0^{t\wedge\tau_m} |\eta_{s-}(y) - \xi_{s-}(y)| ds, \\ \mathcal{R}_5(t\wedge\tau_m) &\le \int_0^{t\wedge\tau_m} \int_{\mathcal{X}\setminus\{0\}} \nu(x) |\rho(x,\eta_{s-},\nu) - \rho(x,\xi_{s-},\nu)| H_2(d\nu) ds.\end{aligned} \tag{27}$$

Taking then expectations in (23) and using the above estimates gives

$$\mathbb{E}[\phi_k(\zeta_{t\wedge\tau_m}(x))] = \mathbb{E}[\phi_k(\zeta_0(x))] + \mathbb{E}\left[\sum_{j=1}^5 \mathcal{R}_j(t\wedge\tau_m)\right]$$

$$\leq \mathbb{E}[\phi_k(\zeta_0(x))] + \mathbb{E}\left[\int_0^{t\wedge\tau_m} |B_0(x,\eta_s) - B_0(x,\xi_s)|ds\right]$$
$$+ C_2(x)\frac{2t}{k} + \frac{C_3(x)t}{k}\int_{\{\|\nu\|\leq 1\}\setminus\{0\}} \nu(x)^2 H_1(x,d\nu)$$
$$+ C_3(x)\left(\int_{\{\|\nu\|>1\}} \nu(x)H_1(x,d\nu)\right)\mathbb{E}\left[\int_0^{t\wedge\tau_m} |\eta_{s-}(x) - \xi_{s-}(x)|ds\right]$$
$$+ \sum_{y\in V\setminus\{x\}} \left(\int_{\mathcal{X}\setminus\{0\}} \nu(x)H_1(y,d\nu)\right) C_3(y)\mathbb{E}\left[\int_0^{t\wedge\tau_m} |\eta_{s-}(y) - \xi_{s-}(y)|ds\right]$$
$$+ \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}} \nu(x)|\rho(x,\eta_{s-},\nu) - \rho(x,\xi_{s-},\nu)|H_2(d\nu)ds\right].$$

Letting first $k\to\infty$, then summing up over x, and finally using (i) yields

$$\mathbb{E}[\|\eta_{t\wedge\tau_m} - \xi_{t\wedge\tau_m}\|] = \sum_{x\in V}\mathbb{E}[|\eta_{t\wedge\tau_m}(x) - \xi_{t\wedge\tau_m}(x)|]$$
$$\leq \sum_{x\in V}\mathbb{E}[|\eta_0(x) - \xi_0(x)|] + \sum_{x\in V} v(x)\mathbb{E}\left[\int_0^{t\wedge\tau_m} |B_0(x,\eta_{s-}) - B_0(x,\xi_{s-})|ds\right]$$
$$+ \sum_{x\in V} C_3(x)\left(\int_{\{\|\nu\|>1\}} \nu(x)H_1(x,d\nu)\right)\mathbb{E}\left[\int_0^{t\wedge\tau_m} |\eta_{s-}(x) - \xi_{s-}(x)|ds\right]$$
$$+ \sum_{x\in V}\sum_{y\in V\setminus\{x\}} \left(\int_{\mathcal{X}\setminus\{0\}} \nu(x)H_1(y,d\nu)\right) C_3(y)\mathbb{E}\left[\int_0^{t\wedge\tau_m} |\eta_{s-}(y) - \xi_{s-}(y)|ds\right]$$
$$+ \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}}\sum_{x\in V} \nu(x)|\rho(x,\eta_{s-},\nu) - \rho(x,\xi_{s-},\nu)|H_2(d\nu)ds\right]$$
$$\leq \mathbb{E}[\|\eta_0 - \xi_0\|] + \mathbb{E}\left[\int_0^{t\wedge\tau_m} \|B_0(\cdot,\eta_{s-}) - B_0(\cdot,\xi_{s-})\|ds\right]$$
$$+ (2C_4 + C_5(m))\mathbb{E}\left[\int_0^{t\wedge\tau_m} \|\eta_{s-} - \xi_{s-}\|ds\right]$$
$$\leq \mathbb{E}[\|\eta_0 - \xi_0\|] + (C_1(m) + 2C_4 + C_5(m))\int_0^t \mathbb{E}\left[\|\eta_{s\wedge\tau_m} - \xi_{s\wedge\tau_m}\|\right] ds$$

where we have used (A1'), (A4), (A5'), and

$$\sum_{x\in V}\sum_{y\in V\setminus\{x\}} \left(\int_{\mathcal{X}\setminus\{0\}} \nu(x)H_1(y,d\nu)\right) C_3(y)\mathbb{E}\left[\int_0^{t\wedge\tau_m} |\eta_{s-}(y) - \xi_{s-}(y)|ds\right]$$
$$= \sum_{y\in V}\left(\int_{\mathcal{X}\setminus\{0\}}\sum_{x\in V\setminus\{y\}} \nu(x)H_1(y,d\nu)\right) C_3(y)\mathbb{E}\left[\int_0^{t\wedge\tau_m} |\eta_{s-}(y) - \xi_{s-}(y)|ds\right]$$
$$\leq C_4\sum_{y\in V}\mathbb{E}\left[\int_0^{t\wedge\tau_m} |\eta_{s-}(y) - \xi_{s-}(y)|ds\right].$$

The assertion of the theorem follows from the Gronwall lemma. Hence it remains to prove the estimates for $\mathcal{R}_j(t\wedge\tau_m)$, $j=1,\dots,5$ in (26). For the first term, we obtain from (A1') combined with (ii) that $\phi_k'(\zeta_s(x))(B_1(x,\eta_s(x)) -$

$B_1(x, \xi_s(x))) \geq 0$ holds a.s. for $s \in [0, t \wedge \tau_m]$. Hence we obtain

$$\mathcal{R}_1(t \wedge \tau_m) = \int_0^{t\wedge\tau_m} \phi_k'(\zeta_s(x))(B_0(x,\eta_s) - B_0(x,\xi_s))ds - \int_0^{t\wedge\tau_m} \phi_k'(\zeta_s(x))(B_1(x,\eta_s(x)) - B_1(x,\xi_s(x)))ds$$
$$\leq \int_0^{t\wedge\tau_m} |B_0(x,\eta_s) - B_0(x,\xi_s)|ds$$

For the second term, we first observe that (A2) and property (iii) yield

$$\phi_k''(\zeta_{s-}(x))|c(x,\eta_{s-}(x)) - c(x,\xi_{s-}(x))| \leq C_2(x)\phi_k''(\zeta_{s-}(x))|\eta_{s-}(x) - \xi_{s-}(x)| \leq C_2(x)\frac{2}{k},$$

where we have used (A2). Hence using the elementary inequality $(a-b)^2 \leq |a^2 - b^2|$ for $a, b > 0$ for the first inequality, we find that

$$\mathcal{R}_2(t \wedge \tau_m) \leq \int_0^{t\wedge\tau_m} \phi_k''(\zeta_s(x))|c(x,\eta_s(x)) - c(x,\xi_s(x))|ds \leq C_2(x)\frac{2t}{k}.$$

To estimate the third term $\mathcal{R}_3$, we decompose the integral against $H_1(x, d\nu)$ into $\{\|\nu\| \leq 1\}\backslash\{0\}$ and $\{\|\nu\| > 1\}$ to find that $\mathcal{R}_3(t) = \mathcal{R}_3^1(t) + \mathcal{R}_3^2(t)$, where

$$\mathcal{R}_3^1(t) = \int_0^t \int_{\{\|\nu\|\leq 1\}\backslash\{0\}} \int_{\mathbb{R}_+} \left(D_{\Delta_0(x,s)}\phi_k(\zeta_{s-}(x)) - \phi_k'(\zeta_{s-}(x))\Delta_0(x,s)\right) dsH_1(x,d\nu)du$$
$$\mathcal{R}_3^2(t) = \int_0^t \int_{\{\|\nu\|>1\}} \int_{\mathbb{R}_+} \left(D_{\Delta_0(x,s)}\phi_k(\zeta_{s-}(x)) - \phi_k'(\zeta_{s-}(x))\Delta_0(x,s)\right) dsH_1(x,d\nu)du.$$

In order to estimate these integrals, we first compute the integral against du. Namely, observe that by (A3), $\zeta_{s-}(x) \leq 0$ implies that $g(x, \eta_{s-}(x)) \leq g(x, \xi_{s-}(x))$ and hence $\Delta_0(x,s) \leq 0$ while $\zeta_{s-}(x) > 0$ implies $\Delta_0(x,s) \geq 0$. Combining both observations we find for $\|\nu\| \leq 1$ by (22)

$$\begin{aligned}
&\int_{\mathbb{R}_+} \left(D_{\Delta_0(x,s)}\phi_k(\zeta_{s-}(x)) - \phi_k'(\zeta_{s-}(x))\Delta_0(x,s)\right) du \\
&= \int_{\mathbb{R}_+} \mathbb{1}_{\{\zeta_{s-}(x)>0\}} \left(D_{\Delta_0(x,s)}\phi_k(\zeta_{s-}(x)) - \phi_k'(\zeta_{s-}(x))\Delta_0(x,s)\right) du \\
&\quad + \int_{\mathbb{R}_+} \mathbb{1}_{\{\zeta_{s-}(x)\leq 0\}} \left(D_{\Delta_0(x,s)}\phi_k(\zeta_{s-}(x)) - \phi_k'(\zeta_{s-}(x))\Delta_0(x,s)\right) du \\
&= \int_{g(x,\xi_{s-}(x))}^{g(x,\eta_{s-}(x))} \mathbb{1}_{\{\zeta_{s-}(x)>0\}} \left(D_{\nu(x)}\phi_k(\zeta_{s-}(x)) - \phi_k'(\zeta_{s-}(x))\nu(x)\right) du \\
&\quad + \int_{g(x,\eta_{s-}(x))}^{g(x,\xi_{s-}(x))} \mathbb{1}_{\{\zeta_{s-}(x)\leq 0\}} \left(D_{-\nu(x)}\phi_k(\zeta_{s-}(x)) + \phi_k'(\zeta_{s-}(x))\nu(x)\right) du \\
&\leq |g(x,\eta_{s-}(x)) - g(x,\xi_{s-}(x))| \frac{\nu(x)^2}{k|\zeta_{s-}(x)|} \leq \frac{C_3(x)}{k}\nu(x)^2
\end{aligned}$$

while for $\|\nu\| > 1$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+} \left(D_{\Delta_0(x,s)}\phi_k(\zeta_{s-}(x)) - \phi_k'(\zeta_{s-}(x))\Delta_0(x,s)\right) du \\
&= \int_{g(x,\xi_{s-}(x))}^{g(x,\eta_{s-}(x))} \mathbb{1}_{\{\zeta_{s-}(x)>0\}} \left(D_{\nu(x)}\phi_k(\zeta_{s-}(x)) - \phi_k'(\zeta_{s-}(x))\nu(x)\right) du \\
&\quad + \int_{g(x,\eta_{s-}(x))}^{g(x,\xi_{s-}(x))} \mathbb{1}_{\{\zeta_{s-}(x)\leq 0\}} \left(D_{-\nu(x)}\phi_k(\zeta_{s-}(x)) + \phi_k'(\zeta_{s-}(x))\nu(x)\right) du \\
&\leq |g(x,\eta_{s-}(x)) - g(x,\xi_{s-}(x))|\, \nu(x) \leq C_3(x)\nu(x)|\eta_{s-}(x) - \xi_{s-}(x)|.
\end{aligned}$$

For the first part, we obtain

$$\mathcal{R}_3^1(t\wedge\tau_m) \leq \frac{C_3(x)t}{k}\int_{\{\|\nu\|\leq 1\}\setminus\{0\}} \nu(x)^2 H_1(x,d\nu),$$

while the second part is estimated as follows:

$$\mathcal{R}_3^2(t\wedge\tau_m) \leq C_3(x)\left(\int_{\{\|\nu\|>1\}} \nu(x)H_1(x,d\nu)\right)\int_0^{t\wedge\tau_m} |\eta_{s-}(x)-\xi_{s-}(x)|ds.$$

For the fourth term, we obtain from (22)

$$\begin{aligned}\mathcal{R}_4(t\wedge\tau_m) &\leq \sum_{y\in V\setminus\{x\}}\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}} \nu(x)|g(y,\eta_{s-}(y))-g(y,\xi_{s-}(y))|dsH_1(y,d\nu)\\ &\leq \sum_{y\in V\setminus\{x\}}\left(\int_{\mathcal{X}\setminus\{0\}}\nu(x)H_1(y,d\nu)\right)C_3(y)\int_0^{t\wedge\tau_m}|\eta_{s-}(y)-\xi_{s-}(y)|ds.\end{aligned}$$

Finally, we find that

$$\begin{aligned}\mathcal{R}_5(t\wedge\tau_m) &\leq \int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}|\Delta_1(x,s)|dsH_2(d\nu)du\\ &= \int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}}\nu(x)|\rho(x,\eta_{s-},\nu)-\rho(x,\xi_{s-},\nu)|dsH_2(d\nu).\end{aligned}$$

This proves the desired inequalities for $\mathcal{R}_j(t\wedge\tau_m)$, $j=1,\ldots,5$ and hence completes the proof of this statement. □

5. Comparison principles

In this section, we show that under conditions (A1) – (A5), the process is monotone with respect to the initial condition. Moreover, we establish a monotonicity principle concerning the drift parameters. Let us start with the following technical result. The desired comparison property is then proved afterwards.

Lemma 5.1. *Suppose that conditions (A1) and (A5) are satisfied. Then*

$$\sum_{x\in V} v(x)(B_0(x,\eta)-B_0(x,\xi))^+ \leq C_1(m)\sum_{x\in V} v(x)(\eta(x)-\xi(x))^+$$

and

$$\begin{aligned}&\int_{\mathcal{X}\setminus\{0\}}\sum_{x\in V} v(x)\nu(x)(\rho(x,\eta,\nu)-\rho(x,\xi,\nu))^+H_2(d\nu)\\ &\qquad \leq C_5(m)\sum_{x\in V} v(x)(\eta(x)-\xi(x))^+\end{aligned}$$

hold for all $\eta,\xi\in\mathcal{X}$ satisfying $\|\eta\|,\|\xi\|\leq m$ for some $m\in\mathbb{N}$ where $z^+=\max\{z,0\}$.

Proof. For given $\eta,\xi\in\mathcal{X}$ we define

$$\min(\eta,\xi)(x) = \begin{cases}\xi(x), & \text{if } \eta(x)\geq\xi(x)\\ \eta(x), & \text{if } \eta(x)<\xi(x),\end{cases} \qquad x\in V.$$

Then $\min(\eta,\xi)\leq\xi$ and $\|\eta-\min(\eta,\xi)\| = \sum_{x\in V} v(x)(\eta(x)-\xi(x))^+$. Using (A1), we obtain for $\eta,\xi\in\mathcal{X}$ satisfying $\|\eta\|,\|\xi\|\leq m$

$$\sum_{x\in V} v(x)(B_0(x,\eta)-B_0(x,\xi))^+$$

$$\le \sum_{x\in V} v(x)(B_0(x,\eta) - B_0(x,\min(\eta,\xi)))^+ + \sum_{x\in V} v(x)(B_0(x,\min(\eta,\xi)) - B_0(x,\xi))^+$$

$$\le \sum_{x\in V} v(x)|B_0(x,\eta) - B_0(x,\min(\eta,\xi))|$$

$$\le C_1(m)\|\eta - \min(\eta,\xi)\| = C_1(m)\sum_{x\in V} v(x)(\eta(x)-\xi(x))^+.$$

Analogously, using (A5), we find that

$$\int_{\mathcal{X}\setminus\{0\}} \sum_{x\in V} v(x)\nu(x)(\rho(x,\eta,\nu) - \rho(x,\xi,\nu))^+ H_2(d\nu)$$

$$\le \int_{\mathcal{X}\setminus\{0\}} \sum_{x\in V} v(x)\nu(x)(\rho(x,\eta,\nu) - \rho(x,\min(\eta,\xi),\nu))^+ H_2(d\nu)$$

$$+ \int_{\mathcal{X}\setminus\{0\}} \sum_{x\in V} v(x)\nu(x)(\rho(x,\min(\eta,\xi),\nu) - \rho(x,\xi,\nu))^+ H_2(d\nu)$$

$$\le \int_{\mathcal{X}\setminus\{0\}} \sum_{x\in V} v(x)\nu(x)|\rho(x,\eta,\nu) - \rho(x,\min(\eta,\xi),\nu)| H_2(d\nu)$$

$$\le C_5(m)\|\eta - \min(\eta,\xi)\| = C_5(m)\sum_{x\in E} v(x)(\eta(x)-\xi(x))^+.$$

This proves the assertion. □

The following is the main result of this section.

Theorem 5.2. *Suppose that conditions (A1) – (A5) are satisfied. Let $(\eta_t)_{t\ge 0}$ and $(\xi_t)_{t\ge 0}$ be two weak solutions to (2) defined on the same stochastic basis. Then for each $m\in\mathbb{N}$ and $t\ge 0$ it holds that*

$$\mathbb{E}\left[\sum_{x\in V} v(x)(\eta_{t\wedge\tau_m}(x) - \xi_{t\wedge\tau_m}(x))^+\right] \le \mathbb{E}\left[\sum_{x\in V} v(x)(\eta_0(x)-\xi_0(x))^+\right] e^{(C_1(m)+2C_4+C_5(m))t},$$

where $\tau_m := \tau_m(\eta)\wedge\tau_m(\xi)$ is a sequence of stopping times with $\tau_m(\eta),\tau_m(\xi)$ defined as in Lemma 3.2. In particular, if $\mathbb{P}[\eta_0\le\xi_0] = 1$, then $\mathbb{P}[\eta_t\le\xi_t,\ \ t\ge 0] = 1$.

Proof. Without loss of generality suppose that $v\equiv 1$, see Lemmma A.1. Define $\zeta_t := \eta_t - \xi_t$ and fix $x\in V$. Let $\phi_k : \mathbb{R}\longrightarrow\mathbb{R}_+$ be a sequence of twice continuously differentiable functions such that for each $k\ge 1$:

(i) $\phi_k(z)\nearrow z^+ := \max\{0,z\}$ as $k\to\infty$ for $z\ge 0$,
(ii) $\phi_k(z) = \phi_k'(z) = \phi_k''(z) = 0$ for $z\le 0$,
(iii) $\phi_k'(z)\in[0,1]$ for $z\ge 0$,
(iv) $\phi_k''(z)z \le 2/k$ holds for all $z\ge 0$.

Note that the sequence in Thm. 4.1 approximates the absolute value function, while the function above approximates the rectified linear unit, which has been previously used in, e.g., [4, 12, 14, 28].

To simplify the notation below, we set $D_h\phi_k(z) := \phi_k(z+h) - \phi_k(z)$ with $z,h\in\mathbb{R}$. Using the mean-value theorem, one can check that (22) holds for all $z,h\in\mathbb{R}$. Applying the Itô formula to $\zeta_t(x)$ gives

$$\phi_k(\zeta_t(x)) = \phi_k(\zeta_0(x)) + \sum_{j=1}^{5}\mathcal{R}_j(t) + \mathcal{M}(t), \tag{28}$$

where the processes $\mathcal{R}_1,\dots,\mathcal{R}_5,\mathcal{M}$ are given as in the proof of Theorem 4.1 and, in particular, also depend on the fixed value x. Let $\tau_m := \tau_m(\eta)\wedge\tau_m(\xi)$ with $\tau_m(\eta),\tau_m(\xi)$ defined in Lemma 3.2. Then $\tau_m\longrightarrow\infty$ and (25) holds. The same arguments as in the appendix prove that $(\mathcal{M}(t\wedge\tau_m))_{t\ge 0}$ is a martingale for each k and each m. Below, we will show that

$$
\begin{aligned}
\mathcal{R}_1(t\wedge\tau_m) &\le \int_0^{t\wedge\tau_m} (B_0(x,\eta_s)-B_0(x,\xi_s))^+ ds\\
\mathcal{R}_2(t\wedge\tau_m) &\le C_2(x)\frac{2t}{k},\\
\mathcal{R}_3(t\wedge\tau_m) &\le \frac{C_3(x)t}{k}\int_{\{\|\nu\|\le 1\}\setminus\{0\}} \nu(x)^2 H_1(x,d\nu)\\
&\qquad + C_3(x)\left(\int_{\{\|\nu\|>1\}} \nu(x)H_1(x,d\nu)\right)\int_0^{t\wedge\tau_m} (\eta_{s-}(x)-\xi_{s-}(x))^+ ds,\\
\mathcal{R}_4(t\wedge\tau_m) &\le \sum_{y\in V\setminus\{x\}}\left(\int_{\mathcal{X}\setminus\{0\}} \nu(x)H_1(y,d\nu)\right) C_3(y)\int_0^{t\wedge\tau_m} (\eta_{s-}(y)-\xi_{s-}(y))^+ ds,\\
\mathcal{R}_5(t\wedge\tau_m) &\le \int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}} \nu(x)(\rho(x,\eta_{s-},\nu)-\rho(x,\xi_{s-},\nu))^+ ds H_2(d\nu)
\end{aligned}
$$

Taking then expectations in (28) and using the above estimates gives

$$
\begin{aligned}
\mathbb{E}[\phi_k(\zeta_{t\wedge\tau_m}(x))] &= \mathbb{E}[\phi_k(\zeta_0(x))] + \mathbb{E}\left[\sum_{j=1}^5 \mathcal{R}_j(t\wedge\tau_m)\right]\\
&\le \mathbb{E}[\phi_k(\zeta_0(x))] + \mathbb{E}\left[\int_0^{t\wedge\tau_m} (B_0(x,\eta_s)-B_0(x,\xi_s))^+ ds\right]\\
&\quad + C_2(x)\frac{2t}{k} + \frac{C_3(x)t}{k}\int_{\{\|\nu\|\le 1\}\setminus\{0\}} \nu(x)^2 H_1(x,d\nu)\\
&\quad + C_3(x)\left(\int_{\{\|\nu\|>1\}} \nu(x)H_1(x,d\nu)\right)\mathbb{E}\left[\int_0^{t\wedge\tau_m} (\eta_{s-}(x)-\xi_{s-}(x))^+ ds\right]\\
&\quad + \sum_{y\in V\setminus\{x\}}\left(\int_{\mathcal{X}\setminus\{0\}} \nu(x)H_1(y,d\nu)\right) C_3(y)\mathbb{E}\left[\int_0^{t\wedge\tau_m} (\eta_{s-}(y)-\xi_{s-}(y))^+ ds\right]\\
&\quad + \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}} \nu(x)(\rho(x,\eta_{s-},\nu)-\rho(x,\xi_{s-},\nu))^+ H_2(d\nu) ds\right].
\end{aligned}
$$

Letting first $k\to\infty$ and then taking the sum over $x\in V$ we get by (i)

$$
\begin{aligned}
\mathbb{E}\left[\sum_{x\in V}(\eta_{t\wedge\tau_m}(x)-\xi_{t\wedge\tau_m}(x))^+\right] &\le \sum_{x\in V}\mathbb{E}\left[(\eta_0(x)-\xi_0(x))^+\right] + \mathbb{E}\left[\int_0^{t\wedge\tau_m}\sum_{x\in V}(B_0(x,\eta_s)-B_0(x,\xi_s))^+ ds\right]\\
&\quad + 2C_4\mathbb{E}\left[\int_0^{t\wedge\tau_m}\sum_{x\in V}(\eta_{s-}(x)-\xi_{s-}(x))^+ ds\right]\\
&\quad + \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}}\sum_{x\in V}\nu(x)(\rho(x,\eta_{s-},\nu)-\rho(x,\xi_{s-},\nu))^+ H_2(d\nu) ds\right]\\
&\le \mathbb{E}\left[\sum_{x\in V}(\eta_0(x)-\xi_0(x))^+\right]\\
&\quad + (C_1(m)+2C_4+C_5(m))\,\mathbb{E}\left[\int_0^{t\wedge\tau_m}\sum_{x\in V}(\eta_{s-}(x)-\xi_s(x))^+ ds\right]
\end{aligned}
$$

where we have used Lemma 5.1. The assertion follows from the Gronwall lemma. Hence it remains to prove the estimates for $\mathcal{R}_j(t \wedge \tau_m)$, $j = 1, \dots, 5$.

The first estimate above follows directly by the properties of ϕ'. Indeed, it follows from (A1) and (ii) that $\phi_k'(\zeta_s(x))(B_1(x, \eta_s(x)) - B_1(x, \xi_s(x))) \geq 0$ holds a.s. for $s \in [0, t \wedge \tau_m]$. Thus we obtain

$$\begin{aligned}
\mathcal{R}_1(t \wedge \tau_m) &= \int_0^{t\wedge\tau_m} \phi_k'(\zeta_{s-}(x))(B(x,\eta_s) - B(x,\xi_s))ds \\
&= \int_0^{t\wedge\tau_m} \phi_k'(\zeta_{s-}(x))(B_0(x,\eta_s) - B_0(x,\xi_s))ds - \int_0^{t\wedge\tau_m} \phi_k'(\zeta_{s-}(x))(B_1(x,\eta_s(x)) - B_1(x,\xi_s(x)))ds \\
&\leq \int_0^{t\wedge\tau_m} \phi_k'(\zeta_{s-}(x))(B_0(x,\eta_s) - B_0(x,\xi_s))ds \\
&\leq \int_0^{t\wedge\tau_m} (B_0(x,\eta_s) - B_0(x,\xi_s))^+ ds.
\end{aligned}$$

The desired estimates for $\mathcal{R}_2, \mathcal{R}_3$, and $\mathcal{R}_4$ can be shown in the same way as in the proof of Theorem 4.1. Let us consider the term $\mathcal{R}_5$. By (ii) we have

$$\begin{aligned}
\int_0^\infty D_{\Delta_1(x,s)}\phi_k(\zeta_{s-}(x))du &\leq \int_0^\infty \mathbb{1}_{\{\rho(x,\eta_{s-},\nu)\geq\rho(x,\xi_{s-},\nu)\}} D_{\Delta_1(x,s)}\phi_k(\zeta_{s-}(x))du \\
&\leq \int_{\rho(x,\xi_{s-},\nu)}^{\rho(x,\eta_{s-},\nu)} \mathbb{1}_{\{\rho(x,\eta_{s-},\nu)\geq\rho(x,\xi_{s-},\nu)\}}\nu(x)du \\
&= (\rho(x,\eta_{s-},\nu) - \rho(x,\xi_{s-},\nu))^+\nu(x).
\end{aligned}$$

This implies the desired estimate for $\mathcal{R}_5(t \wedge \tau_m)$. Hence, we have shown all the desired inequalities for $\mathcal{R}_j(t \wedge \tau_m)$, $j = 1, \dots, 5$ and the proof of the theorem is complete. □

Theorem 5.2 can be generalized to the case when $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ are solutions to equations with different functions B, B_0, B_1, c, g, ρ. Here, we only give a simple version with different drifts.

Corollary 5.3. *Let $(B, B_0, B_1, c, g, \rho)$ and $(\widetilde{B}, \widetilde{B}_0, \widetilde{B}_1, c, g, \rho)$ be $C_{\overline{1,6}}$-admissible tuples. On the same stochastic basis let $(\eta_t)_{t\geq 0}$ satisfy (2) and let $(\xi_t)_{t\geq 0}$ satisfy*

$$\begin{aligned}
\xi_t(x) = \xi_0(x) &+ \int_0^t \widetilde{B}(x,\xi_s)ds + \int_0^t \sqrt{2c(x,\xi_s(x))}dW_s(x) \\
&+ \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x)\mathbb{1}_{\{u\leq g(x,\xi_{s-}(x))\}}\widetilde{N}_x(ds,d\nu,du) \\
&+ \sum_{y\in V\setminus\{x\}} \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x)\mathbb{1}_{\{u\leq g(y,\xi_{s-}(y))\}}N_y(ds,d\nu,du) \\
&+ \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x)\mathbb{1}_{\{u\leq\rho(x,\xi_{s-},\nu)\}}M(ds,d\nu,du).
\end{aligned} \tag{29}$$

Assume that for all $\alpha, \beta \in \mathcal{X}$, $\alpha \leq \beta$, and $x \in V$ one has $B(x,\alpha) \leq \widetilde{B}(x,\beta)$. Then for each $m \in \mathbb{N}$ and $t \geq 0$ it holds that

$$\mathbb{E}\left[\sum_{x\in V} v(x)(\eta_{t\wedge\tau_m}(x) - \xi_{t\wedge\tau_m}(x))^+\right] \leq \mathbb{E}\left[\sum_{x\in V} v(x)(\eta_0(x) - \xi_0(x))^+\right] e^{(C_1(m)+2C_4+C_5(m))t},$$

where $\tau_m := \tau_m(\eta) \wedge \tau_m(\xi)$ is a sequence of stopping times with $\tau_m(\eta), \tau_m(\xi)$ defined as in Lemma 3.2. In particular, if $\mathbb{P}[\eta_0 \leq \xi_0] = 1$, then $\mathbb{P}[\eta_t \leq \xi_t,\ t \geq 0] = 1$.

Proof. The proof follows the same steps as the proof of Theorem 5.2, the only difference is that with these settings

$$\mathcal{R}_1(t) = \int_0^t \phi_k'(\zeta_s(x))\left(B(x,\eta_s) - \widetilde{B}(x,\xi_s)\right)ds,$$

so we only need to note that

$$\int_0^t \phi_k'(\zeta_s(x))\left(B(x,\eta_s)-\widetilde{B}(x,\xi_s)\right)ds \le \int_0^t \phi_k'(\zeta_s(x))\left(B(x,\eta_s)-B(x,\xi_s)\right)ds.$$

□

Finally, we formulate an auxiliary comparison principle used for the construction of solutions of (2).

Theorem 5.4. *Let $|V|<\infty$ and let $(\widetilde{B},\widetilde{B}_0,\widetilde{B}_1,c,\widetilde{g},\widetilde{\rho})$ be a $C_{\overline{1,6}}$-admissible tuple and for $V'\subset V$ let*

$$B(x,\alpha)=\widetilde{B}(x,\alpha)\mathbb{1}\{x\in V'\} \tag{30}$$

$$g(x,\alpha(x))=\widetilde{g}(x,\alpha(x))\mathbb{1}\{x\in V'\}, \tag{31}$$

$$\rho(x,\alpha,\nu)=\widetilde{\rho}(x,\alpha,\nu)\mathbb{1}\{x\in V'\}. \tag{32}$$

On the same stochastic basis let $(\eta_t)_{t\ge 0}$ satisfy (2) and let $(\xi_t)_{t\ge 0}$ satisfy

$$\begin{aligned}\xi_t(x)&=\xi_0(x)+\int_0^t\widetilde{B}(x,\xi_s)ds+\int_0^t\sqrt{2c(x,\xi_s(x))}dW_s(x)\\&+\int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\le\widetilde{g}(x,\xi_{s-}(x))\}}\widetilde{N}_x(ds,d\nu,du)\\&+\sum_{y\in V\setminus\{x\}}\int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\le\widetilde{g}(y,\xi_{s-}(y))\}}N_y(ds,d\nu,du)\\&+\int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\le\widetilde{\rho}(x,\xi_{s-},\nu)\}}M(ds,d\nu,du),\end{aligned}\tag{33}$$

Assume further that $\mathbb{P}[\eta_0\le\xi_0]=1$ and for $x\in V\setminus V'$, $\mathbb{P}[\eta_0(x)=0]=1$. Then $\mathbb{P}[\eta_t\le\xi_t,\ t\ge 0]=1$.

Proof. Since $(\widetilde{B},\widetilde{B}_0,\widetilde{B}_1,c,\widetilde{g},\widetilde{\rho})$ is $C_{\overline{1,6}}$-admissible, the tuple (B,B_0,B_1,c,g,ρ) is a $C_{\overline{1,6}}$-admissible as well by (30)-(32). For $x\in V\setminus V'$ we have a.s. $\eta_t(x)=0$ and hence $\mathbb{P}[\eta_t(x)\le\xi_t(x),\ t\ge 0]=\mathbb{P}[0\le\xi_t(x),\ t\ge 0]=1$. Let $\{\phi_k\}_{k\in\mathbb{N}}$ be the sequence of functions introduced in the proof of Theorem 5.2. Set $\zeta_t:=\eta_t-\xi_t$. Recall the notation $D_h\phi_k(z):=\phi_k(z+h)-\phi_k(z)$ for $z,h\in\mathbb{R}$.

For $x\in V'$ by the the Itô formula

$$\phi_k(\zeta_t(x))=\phi_k(\zeta_0(x))+\sum_{j=1}^5\mathcal{D}_j(t)+\mathcal{M}(t), \tag{34}$$

where

$$\begin{aligned}\mathcal{D}_1(t)&=\int_0^t\phi_k'(\zeta_s(x))\left(B(x,\eta_s)-\widetilde{B}(x,\xi_s)\right)ds\\ \mathcal{D}_2(t)&=\frac{1}{2}\int_0^t\phi_k''(\zeta_s(x))\left(\sqrt{2c(x,\eta_s(x))}-\sqrt{2c(x,\xi_s(x))}\right)^2ds\\ \mathcal{D}_3(t)&=\int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\left(D_{\widetilde{\Delta}_0(x,s)}\phi_k(\zeta_{s-}(x))-\phi_k'(\zeta_{s-}(x))\widetilde{\Delta}_0(x,s)\right)dsH_1(x,d\nu)du\\ \mathcal{D}_4(t)&=\sum_{y\in V\setminus\{x\}}\int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}D_{\widetilde{\Delta}_0(y,s)}\phi_k(\zeta_{s-}(x))dsH_1(y,d\nu)du\\ \mathcal{D}_5(t)&=\int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}D_{\widetilde{\Delta}_1(x,s)}\phi_k(\zeta_{s-}(x))dsH_2(d\nu)du\end{aligned}$$

with increments given by

$$\widetilde{\Delta}_0(z,s) = \nu(x)\left(\mathbb{1}_{\{u\le g(z,\eta_{s-}(z))\}} - \mathbb{1}_{\{u\le \widetilde{g}(z,\xi_{s-}(z))\}}\right),$$
$$\widetilde{\Delta}_1(z,s) = \nu(z)\left(\mathbb{1}_{\{u\le \rho(z,\eta_{s-},\nu)\}} - \mathbb{1}_{\{u\le \widetilde{\rho}(z,\xi_{s-},\nu)\}}\right)$$

and

$$\begin{aligned}\mathcal{M}(t) = &\int_0^t \phi_k'(\zeta_s(x))\left(\sqrt{2c(x,\eta_s(x))} - \sqrt{2c(x,\xi_s(x))}\right) dW_s(x)\\ &+ \sum_{y\in V}\int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+} D_{\widetilde{\Delta}_0(y,s)}\phi_k(\zeta_{s-}(x))\widetilde{N}_y(ds,d\nu,du)\\ &+ \int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+} D_{\widetilde{\Delta}_1(x,s)}\phi_k(\zeta_{s-}(x))\widetilde{M}(ds,d\nu,du).\end{aligned}$$

The process $(\mathcal{M}(t), t\ge 0)$ is a local martingale. For $x\in V'$, we use (30)-(32) to find that

$$\mathcal{D}_j(t) = \mathcal{R}_j(t), \quad t\ge 0, j = 1,2,3,5. \tag{35}$$

where $\mathcal{R}_j(t)$ are given as in the proof of Theorem 4.1. For $\mathcal{D}_4(t)$ we write

$$\begin{aligned}\mathcal{D}_4(t) = &\sum_{y\in V\setminus V'}\int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\left[\phi_k(\zeta_{s-}(x) + \widetilde{\Delta}_0(y,s)) - \phi_k(\zeta_{s-}(x))\right] ds H_1(y,d\nu)du\\ &+ \sum_{y\in V'\setminus\{x\}}\int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\left[\phi_k(\zeta_{s-}(x) + \widetilde{\Delta}_0(y,s)) - \phi_k(\zeta_{s-}(x))\right] ds H_1(y,d\nu)du\end{aligned}$$

Recall that Δ_0, Δ_1 were introduced in the proof of Theorem 4.1. For $y\in V\setminus V'$ for $u>0$

$$\widetilde{\Delta}_0(y,s) = -\nu(x)\mathbb{1}_{\{u\le \widetilde{g}(y,\xi_{s-}(y))\}} \le 0 = \Delta_0(y,s)$$

whereas for $y\in V'\setminus\{x\}$ we have $\widetilde{\Delta}_0(y,s) = \Delta_0(y,s)$. Since ϕ_k is non-decreasing, we arrive at

$$\begin{aligned}\mathcal{D}_4(t) \le &\sum_{y\in V\setminus V'}\int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\left[\phi_k(\zeta_{s-}(x) + \Delta_0(y,s)) - \phi_k(\zeta_{s-}(x))\right] ds H_1(y,d\nu)du\\ &+ \sum_{y\in V'\setminus\{x\}}\int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\left[\phi_k(\zeta_{s-}(x) + \Delta_0(y,s)) - \phi_k(\zeta_{s-}(x))\right] ds H_1(y,d\nu)du\\ &= \mathcal{R}_4(t).\end{aligned}$$

Combining this with (34) and (35) we get

$$\phi_k(\zeta_t(x)) \le \phi_k(\zeta_0(x)) + \sum_{j=1}^5 \mathcal{R}_j(t) + \mathcal{M}(t).$$

From here, the proof goes in the same path as the proof of Theorem 5.2 follows from (28). The fact that here we have an inequality instead of an equality in (28) does not make a difference. □

6. Construction of a weak solution

Firstly, we study the case where V is finite. In such a case $\mathcal{X} = \mathbb{R}_+^{|V|}$ and we take $v(x) = 1$ so that $\|\eta\| = \sum_{k=1}^{|V|}|\eta_k|$ corresponds to the 1-norm on $\mathbb{R}^{|V|}$. In this case, (2) becomes a classical SDE for which we may use existing results on the existence of weak solutions. The precise statement is summarized in the next lemma.

Lemma 6.1. *Suppose that V is a finite set and that conditions (A1) – (A6) are satisfied for $v(x)=1$. Then for each η_0 being $\mathcal{F}_0$-measurable with $\mathbb{E}[\|\eta_0\|]<\infty$, (2) has a unique strong solution in $\mathcal{X}=\mathbb{R}_+^{|V|}$.*

Proof. It follows from [1] that for each $n\geq 1$ the equation

$$\begin{aligned}
\eta_t(x) &= \eta_0(x)+\int_0^t B(x,\eta_s^+)ds+\int_0^t\sqrt{2c(x,\eta_s^+(x))}dW_s(x)\\
&\quad+\int_0^t\int_{\mathcal{X}\backslash\{0\}}\int_{\mathbb{R}_+}\mathbb{1}_{\{\|\nu\|\leq n\}}\nu(x)\mathbb{1}_{\{u\leq g(x,\eta_{s-}^+(x))\}}\widetilde{N}_x(ds,d\nu,du)\\
&\quad+\sum_{y\in V\backslash\{x\}}\int_0^t\int_{\mathcal{X}\backslash\{0\}}\int_{\mathbb{R}_+}\mathbb{1}_{\{\|\nu\|\leq n\}}\nu(x)\mathbb{1}_{\{u\leq g(y,\eta_{s-}^+(y))\}}N_y(ds,d\nu,du)\\
&\quad+\int_0^t\int_{\mathcal{X}\backslash\{0\}}\int_{\mathbb{R}_+}\mathbb{1}_{\{\|\nu\|\leq n\}}\nu(x)\mathbb{1}_{\{u\leq\rho(x,\eta_{s-}^+,\nu)\}}M(ds,d\nu,du)
\end{aligned}\tag{36}$$

with $\eta^+(y)=\max\{0,\eta(y)\}$ for $y\in V$ has a weak solution on $\mathcal{X}=\mathbb{R}^{|V|}$. Since the coefficients $c^+(x,t)=c(x,t^+)$, $g^+(x,t)=g(x,t^+)$, and $\rho^+(x,t,\nu)=\rho(x,t^+,\nu)$ still satisfy the conditions (A1) – (A6), in view of Theorem 4.1, also pathwise uniqueness holds and hence this solution is strong. We prove that this solution is nonnegative by following the argument given in [14, Section 2]. Suppose that there exists $\varepsilon>0$ and $x\in V$ such that $\tau=\inf\{t\geq 0\ :\ \eta_t(x)\leq -\varepsilon\}$ satisfies $\mathbb{P}[\tau<\infty]>0$. Then $\eta_\tau(x)=\eta_{\tau-}(x)\leq-\varepsilon$ holds on $\{\tau<\infty\}$. Let $\sigma=\inf\{s\in(0,\tau)\ :\ \eta_t(x)\leq 0,\ \forall t\in[s,\tau]\}$. Then $\sigma<\tau$ a.s., we can find a deterministic time $r\geq 0$ such that $\{\sigma\leq r<\tau\}$ has positive probability. A.s. on this event, we find for $t\geq r$

$$\begin{aligned}
\eta_{t\wedge\tau}(x) &= \eta_{r\wedge\tau}(x)+\int_{r\wedge\tau}^{t\wedge\tau}B(x,\eta_s^+)ds+\sum_{y\in E\backslash\{x\}}\int_{r\wedge\tau}^{t\wedge\tau}\int_{\mathcal{X}\backslash\{0\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\leq g(y,\eta_{s-}^+(y))\}}N_y(ds,d\nu,du)\\
&\quad+\int_{r\wedge\tau}^{t\wedge\tau}\int_{\mathcal{X}\backslash\{0\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\leq\rho(x,\eta_{s-}^+,\nu)\}}M(ds,d\nu,du).
\end{aligned}$$

In view of condition (A1) we have $B(x,\eta)\geq 0$ whenever $\eta\in\mathcal{X}$ is such that $\eta(x)=0$. Thus $t\longmapsto\eta_{t\wedge\tau}(x)$ is non-decreasing. Since $\eta_r(x)>-\varepsilon$ on $\{r<\tau\}$, we get a contradiction to $\eta_\tau(x)=\eta_{\tau-}(x)\leq-\varepsilon$. Hence, the solution is nonnegative.

It remains to show that we can pass to the limit $n\to\infty$. This procedure is standard, so we only provide a proof sketch. Let $(\eta_t^n)_{n\geq 1}$ be the unique strong solution of (36). It is not difficult to show that

$$\sup_{n\geq 1}\mathbb{E}\left[\sup_{t\in[0,T]}|\eta_t^{(n)}(x)|\right]<\infty,\qquad\forall x\in V.$$

Hence, using the Aldous criterion, we find that the sequence of processes $(\eta_t^n)_{n\geq 1}$ is tight on the Skorohod space. Using convergence of the martingale problems, we may show that any of its limits is a weak solution of (2) (with $|V|<\infty$). This completes the proof. $\square$

In the second step, we use Lemma 6.1 to approximate a weak solution via $V_N\nearrow V$ where V_N is an increasing sequence of finite sets.

Theorem 6.2. *Suppose that conditions (A1) – (A6) are satisfied. Then weak existence holds for (2) and any $\mathcal{F}_0$-measurable initial condition $\eta_0\in\mathcal{X}$ satisfying $\mathbb{E}[\|\eta_0\|]<\infty$.*

Proof. *Step 1.* Fix any stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbb{P})$, let noise terms given as in (N1) – (N4), and let η_0 be an $\mathcal{F}_0$-measurable random variable with $\mathbb{E}[\|\eta_0\|]<\infty$. Let $(V_N)_{N\in\mathbb{N}}$ be a sequence of finite sets in V such that $V_N\nearrow V$. Define $B^N(x,\eta)=\mathbb{1}_{V_N}(x)B_0(x,\eta)-B_1(x,\eta(x))$, $g^N(x,t)=\mathbb{1}_{V_N}(x)g(x,t)$, and $\rho^N(x,\eta,\nu)=\mathbb{1}_{V_N}(x)\rho(x,\eta,\nu)$. Then conditions (A1) – (A6) are still satisfied with V replaced by V_N, and (2) takes for these restricted coefficients the form

$$\eta_t^N(x)=\eta_0^N(x)+\int_0^t B^N(x,\eta_s^N)ds+\int_0^t\sqrt{2c(x,\eta_s^N(x))}dW_s(x)$$

$$+\int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x) \mathbb{1}_{\{u \le g^N(x, \eta_{s-}^N(x))\}} \widetilde{N}_x(ds, d\nu, du)$$

$$+ \sum_{y \in V\setminus\{x\}} \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x) \mathbb{1}_{\{u \le g^N(y, \eta_{s-}^N(y))\}} N_y(ds, d\nu, du)$$

$$+\int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x) \mathbb{1}_{\{u \le \rho^N(x, \eta_{s-}^N, \nu)\}} M(ds, d\nu, du)$$

where η_0^N is defined by $\eta_0^N(x) = \mathbb{1}_{V_N}(x)\eta_0(x)$. Thus, the equation is effectively an equation for $\eta_t^N(x)$ with $x \in V_N$, which has a unique, strong solution due to Lemma 6.1.

Step 2. Using $B^N \le B^{N+1}$ and $\rho^N \le \rho^{N+1}$ and the comparison principle in Theorem 5.4, we find that $\mathbb{P}[\eta_t^N \le \eta_t^{N+1},\ t \ge 0] = 1$ for $N \ge 1$. Since $B^N \le B$ and $\rho^N \le \rho$ for all $N \in \mathbb{N}$, we may apply Theorem 3.3 with C independent of N to show that

$$\sup_{t\in[0,T]} \sup_{N\ge 1} \mathbb{E}[\|\eta_t^N\|] < \infty, \qquad T > 0.$$

Define a new process $(\eta_t)_{t\ge0}$ by $\eta_t(x) = \sup_{N\ge1} \eta_t^N(x)$ for $x \in V$. Then $(\eta_t)_{t\ge0}$ is $(\mathcal{F}_t)_{t\ge0}$-adapted and by monotone convergence, we see that

$$\sup_{t\in[0,T]} \mathbb{E}[\|\eta_t\|] \le \sup_{t\in[0,T]} \sup_{N\ge1} \mathbb{E}[\|\eta_t^N\|] < \infty,$$

i.e., $(\eta_t)_{t\ge0}$ takes values in $\mathcal{X}$. Note that by monotone convergence, we also have

$$\lim_{N\to\infty} \int_0^T \mathbb{E}\left[\|\eta_t - \eta_t^N\|\right] dt = 0. \tag{37}$$

Step 3. The arguments in Step 2 already infer that the process $(\eta_t)_{t\ge0}$ is $\mathcal{X}$-valued. Therefore, it remains to show that $(\eta_t)_{t\ge0}$ is a solution to (2). We consider all terms of (2) separately. Convergence of the initial conditions, i.e. $\lim_{N\to\infty} \eta_0^N(x) = \eta_0(x)$ is clear. For the drift, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \mathbb{1}_{V_N}(x) B(x, \eta_s^N) ds - \int_0^t B(x, \eta_s) ds\right|\right] \\
&\le \int_0^t \mathbb{E}\left[|B_0(x, \eta_s^N) - B_0(x, \eta_s)|\right] ds + \int_0^t \mathbb{E}\left[|B_1(x, \eta_s^N(x)) - B_1(x, \eta_s(x))|\right] ds \\
&\quad + \mathbb{1}_{V_N^c}(x) \int_0^t \mathbb{E}\left[|B(x, \eta_s)|\right] ds.
\end{aligned}$$

The first two terms converge to zero as $N \to \infty$ by monotone convergence being applicable due to condition (A1). The last term is finite due to the boundedness of the first moment and hence converges to zero for fixed $x \in V$. For the continuous noise part, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \sqrt{2c(x, \eta_s(x))} dW_s(x) - \int_0^t \sqrt{2c(x, \eta_s^N(x))} dW_s(x)\right|^2\right] \\
&= \int_0^t \mathbb{E}\left[\left(\sqrt{2c(x, \eta_s(x))} - \sqrt{2c(x, \eta_s^N(x))}\right)^2\right] ds \\
&\le 2\int_0^t \mathbb{E}\left[|c(x, \eta_s(x)) - \mathbb{1}_{V_N}(x) c(x, \eta_s^N(x))|\right] ds \\
&\le 2\int_0^t \mathbb{E}\left[1_{\{\eta_s(x)\le R\}} |c(x, \eta_s(x)) - c(x, \eta_s^N(x))|\right] ds + 2\int_0^t \mathbb{E}\left[\mathbb{1}_{\{\eta_s(x)>R\}} |c(x, \eta_s(x)) - c(x, \eta_s^N(x))|\right] ds \\
&\le 2C_2(x) \mathbb{E}\left[|\eta_s(x) - \eta_s^N(x)|\right] ds + 4C_2(x) \int_0^t \mathbb{E}\left[1_{\{\eta_s(x)>R\}} \eta_s(x)\right] ds
\end{aligned}$$

where we have used $\eta_s^N(x) \le \eta_s(x)$. Thus, taking first for fixed R the limit $N \to \infty$, then letting $R \to \infty$, proves the convergence to zero.

For the stochastic integrals against $\widetilde{N}_x$, we split the integrals into $\{\|\nu\| \le 1\} \setminus \{0\}$ and $\{\|\nu\| > 1\}$ and study them separately. First, note that using Itô's isometry (e.g. [17, p. 63]),

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^t \int_{\{\|\nu\|\le 1\}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x)\mathbb{1}_{\{u\le g(x,\eta_{s-}(x))\}}\tilde{N}_x(ds,d\nu,du) \\
&\qquad - \int_0^t \int_{\{\|\nu\|\le 1\}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x)\mathbb{1}_{\{u\le g(x,\eta^N_{s-}(x))\mathbb{1}_{V_N}(x)\}}\tilde{N}_x(ds,d\nu,du)\Big)^2\Big] \\
&= \int_0^t \int_{\{\|\nu\|\le 1\}\setminus\{0\}} \nu(x)^2 \mathbb{E}\Big[\int_{\mathbb{R}_+} |\mathbb{1}_{\{u\le g(x,\eta_{s-}(x))\}} - \mathbb{1}_{\{u\le g(x,\eta^N_{s-}(x))\}}\mathbb{1}_{V_N}(x)|du\Big] H_1(x,d\nu)ds \\
&\le \mathbb{1}_{V_N}(x)\int_0^t \int_{\{\|\nu\|_V\le 1\}\setminus\{0\}} \nu(x)^2 \mathbb{E}|g(x,\eta_{s-}(x)) - g(x,\eta^N_{s-}(x))| H_1(x,d\nu)ds \\
&\qquad + \mathbb{1}_{E_N^c}(x)\int_0^t \int_{\{\|\nu\|\le 1\}\setminus\{0\}} \nu(x)^2 \mathbb{E}[g(x,\eta_{s-}(x)] H_1(x,d\nu)ds \\
&\le C_3(x)\int_0^t \int_{\{\|\nu\|\le 1\}\setminus\{0\}} \nu(x)^2 \mathbb{E}|\eta_{s-}(x) - \eta^N_{s-}(x)| H_1(x,d\nu) \\
&\qquad + \mathbb{1}_{V_N^c}(x)C_3(x)\int_0^t \int_{\{\|\nu\|\le 1\}\setminus\{0\}} \nu(x)^2 \mathbb{E}(\eta_{s-}(x)) H_1(x,d\nu)ds
\end{aligned}$$

where we used (A3) in the end. The first term tends to zero as $N \to \infty$ due to (37), while the second one is due to the indicator function. For the integrals against $\{\|\nu\| > 1\}$ we find that

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big[\Big|\int_0^t \int_{\{\|\nu\|>1\}} \int_{\mathbb{R}_+} \nu(x)\mathbb{1}_{\{u\le g(x,\eta_s(x))\}}\tilde{N}_x(ds,d\nu,du) \\
&\qquad - \int_0^t \int_{\{\|\nu\|>1\}} \int_{\mathbb{R}_+} \nu(x)\mathbb{1}_{\{u\le g(x,\eta^N_s(x))\}}\mathbb{1}_{V_N}(x)\tilde{N}_x(ds,d\nu,du)\Big|\Big] \\
&\le \int_0^t \int_{\{\|\nu\|>1\}} \nu(x)\mathbb{E}\Big[\int_{\mathbb{R}_+} \big|\mathbb{1}_{\{u\le g(x,\eta_s(x))\}} - \mathbb{1}_{\{u\le g(x,\eta^N_s(x))\}}\mathbb{1}_{V_N}(x)\big|du\Big] H_1(x,d\nu)ds \\
&\le \mathbb{1}_{V_N^c}(x)\int_0^t \int_{\{\|\nu\|>1\}} \nu(x)\mathbb{E}\big[g(x,\eta_s(x))\big] H_1(x,d\nu)ds \\
&\quad + \mathbb{1}_{V_N}(x)\int_0^t \int_{\{\|\nu\|>1\}} \nu(x)\mathbb{E}\big[\big|g(x,\eta_s(x)) - g(x,\eta^N_s(x))\big|\big] H_1(x,d\nu)ds \\
&= \mathbb{1}_{V_N^c}(x)C_3(x)\int_{\{\|\nu\|>1\}} \nu(x)H_1(x,d\nu)\int_0^t \mathbb{E}[\eta_s(x)]ds \\
&\quad + C_3(x)\int_{\{\|\nu\|>1\}} \nu(x)H_1(x,d\nu)\int_0^t \mathbb{E}\big|\eta_s(x) - \eta^N_s(x)\big|ds
\end{aligned}$$

Also, the right-hand side tends to zero as $N \to \infty$.

For the integrals against N_y, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\sum_{y\in E\setminus\{x\}} \int_0^t \int_{\mathcal{X}\setminus\{0\}} \int_{\mathbb{R}_+} \nu(x)\mathbb{1}_{\{u\le g(y,\eta_{s-}(y))\}} - \nu(x)\mathbb{1}_{\{u\le g(y,\eta^N_{s-}(y))\}}\mathbb{1}_{V_N}(x)N_y(ds,d\nu,du)\Big|\Big] \\
&\le \mathbb{1}_{V_N^c}(x)\sum_{y\in V\setminus\{x\}} C_3(y)\int_{\mathcal{X}\setminus\{0\}} \nu(x)H_1(y,d\nu)\int_0^t \mathbb{E}[\eta_s(y)]ds
\end{aligned}$$

$$+\mathbb{1}_{V_N}(x)\sum_{y\in V\setminus\{0\}}C_3(y)\int_{\mathcal{X}\setminus\{0\}}\nu(x)H_1(y,d\nu)\int_0^t\mathbb{E}\big[|\eta_s(y)-\eta_s^N(y)|\big]ds$$

We estimate both terms separately using (A4). The first one is bounded by

$$\mathbb{1}_{V_N^c}(x)\sum_{y\in E\setminus\{x\}}C_3(y)\int_{\mathcal{X}\setminus\{0\}}\nu(x)H_1(y,d\nu)\int_0^t\mathbb{E}[\eta_s(y)]ds\le\mathbb{1}_{V_N^c}(x)\frac{C_4}{v(x)}\sum_{y\in V\setminus\{x\}}v(y)\int_0^t\mathbb{E}[\eta_s(y)]ds$$

$$=\mathbb{1}_{V_N^c}(x)\frac{C_4}{v(x)}\int_0^t\mathbb{E}[\|\eta_s\|]ds<\infty.$$

Since the last expression is finite, it tends to zero as $N\to\infty$. The second term tends to zero as $N\to\infty$ due to (37) and

$$\sum_{y\in V\setminus\{x\}}C_3(y)\int_{\mathcal{X}\setminus\{0\}}\nu(x)H_1(y,d\nu)\int_0^t\mathbb{E}\big[|\eta_s(y)-\eta_s^N(y)|\big]ds\le\frac{C_4}{v(x)}\int_0^t\mathbb{E}\big[\|\eta_s-\eta_s^N\|\big]ds.$$

Finally, for the last integral, we find that

$$\mathbb{E}\Big[\Big|\int_0^t\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\le\rho(x,\eta_{s-},\nu)\}}-\nu(x)\mathbb{1}_{\{u\le\rho(x,\eta_{s-}^N,\nu)\}}\mathbb{1}_{V_N}(x)M(ds,d\nu,du)\Big|\Big]$$

$$\le\mathbb{1}_{V_N^c}(x)\mathbb{E}\Big[\int_0^t\int_{\mathcal{X}\setminus\{0\}}\nu(x)\rho(x,\eta_{s-},\nu)dsH_2(d\nu)\Big]$$

$$+\mathbb{1}_{V_N}(x)\mathbb{E}\Big[\int_0^t\int_{\mathcal{X}\setminus\{0\}}\nu(x)|\rho(x,\eta_{s-},\nu)-\rho(x,\eta_{s-}^N,\nu)|dsH_2(d\nu)\Big].$$

By (A6), the first term is finite (and hence convergent) due to

$$\mathbb{E}\Big[\int_0^t\int_{\mathcal{X}\setminus\{0\}}\nu(x)\rho(x,\eta_{s-},\nu)dsH_2(d\nu)\Big]\le\frac{C_6}{v(x)}\int_0^t(1+\mathbb{E}\|\eta_s\|)ds<\infty.$$

For the second term, fix $R>0$. Using (A5) and (A6), we get

$$\mathbb{E}\Big[\int_0^t\int_{\mathcal{X}\setminus\{0\}}\nu(x)|\rho(x,\eta_{s-},\nu)-\rho(x,\eta_{s-}^N,\nu)|dsH_2(d\nu)\Big]$$

$$\le\int_0^t\int_{\mathcal{X}\setminus\{0\}}\nu(x)\mathbb{E}\big[\mathbb{1}_{\{\|\eta_{s-}\|\le R\}}|\rho(x,\eta_{s-},\nu))-\rho(x,\eta_{s-}^N,\nu)|H_2(d\nu)ds$$

$$+\int_0^t\mathbb{E}\Big[\int_{\mathcal{X}\setminus\{0\}}\nu(x)\mathbb{1}_{\{\|\eta_{s-}\|>R\}}|\rho(x,\eta_{s-},\nu)-\rho(x,\eta_{s-}^N,\nu)|H_2(d\nu)\Big]ds$$

$$\le\frac{C_5(R)}{v(x)}\int_0^t\mathbb{E}\|\eta_{s-}-\eta_{s-}^N\|_V ds+\int_0^t\mathbb{E}\Big[\int_{\mathcal{X}\setminus\{0\}}\nu(x)\mathbb{1}_{\{\|\eta_s\|>R\}}\rho(x,\eta_{s-},\nu)H_2(d\nu)\Big]ds$$

$$\le\frac{C_5(R)}{v(x)}\int_0^t\mathbb{E}\|\eta_{s-}-\eta_{s-}^N\|ds+\int_0^t\frac{C_6}{v(x)}\mathbb{E}\big[\mathbb{1}_{\{\|\eta_s\|>R\}}(1+\|\eta_{s-}\|)\big]ds.$$

As in the calculation for the Brownian part, this estimate implies convergence to zero when $N\to\infty$. □

7. Proof of Theorem 1.2

In this section, we prove the existence of an invariant measure and convergence in the Wasserstein distance towards this measure, i.e., we prove Theorem 1.2.

Proof of Theorem 1.2. Let $(\eta_t)_{t\ge0}$ and $(\xi_t)_{t\ge0}$ be the unique solutions to (2) with deterministic initial conditions $\eta_0,\xi_0\in\mathcal{X}$ such that $\xi_0\le\eta_0$. Again, without loss of generality we suppose that $v\equiv1$, see Lemma A.1. Then $\xi_t\le\eta_t$

a.s., and hence

$$\begin{aligned}\mathbb{E}[\|\eta_t-\xi_t\|] &= \sum_{x\in V}\mathbb{E}[(\eta_t(x)-\xi_t(x))] \\ &= \sum_{x\in V}(\mathbb{E}[\eta_t(x)]-\mathbb{E}[\xi_t(x)]) \\ &= \sum_{x\in V}(\eta_0(x)-\xi_0(x)) + \sum_{x\in V}\int_0^t \mathbb{E}\left[\widetilde{B}(x,\eta_s(x))-\widetilde{B}(x,\xi_s(x)\right] ds \\ &\le \sum_{x\in V}(\eta_0(x)-\xi_0(x)) - A\int_0^t\left(\sum_{x\in V}\mathbb{E}[(\eta_s(x)-\xi_s(x))]\right) ds \\ &= \|\eta_0-\xi_0\| - A\int_0^t \mathbb{E}[\|\eta_s-\xi_s\|]ds.\end{aligned}$$

The Gronwall lemma yields $\mathbb{E}[\|\eta_t-\xi_t\|]\le \mathbb{E}[\|\eta_0-\xi_0\|]e^{-At}$. For general deterministic $\xi_0,\eta_0\in V$ we let $V=\{x_k : k\ge 1\}$ be a numeration of V, and define

$$\xi_0^n(x) = \begin{cases}\eta_0(x_k), & k=1,\dots,n\\ \xi_0(x_k), & k>n\end{cases}$$

with $\xi_0^0=\xi_0$. Then

$$\xi_0^{n+1}(x_k)-\xi_0^n(x_k) = \begin{cases}0, & k\ne n+1\\ \eta_0(x_{n+1})-\xi_0(x_{n+1}), & k=n+1\end{cases}$$

and hence for each $n\in\mathbb{N}$ either $\xi_0^n\le\xi_0^{n+1}$ or $\xi_0^{n+1}\le\xi_0^n$. Let $(\xi_t^n)_{t\ge0}$ be the unique solution of (2) with initial condition ξ_0^n. Previous consideration yields

$$\mathbb{E}[\|\xi_t^n-\xi_t^{n+1}\|]\le\mathbb{E}[\|\xi_0^n-\xi_0^{n+1}\|]e^{-At} = |\eta_0(x_{n+1})-\xi_0(x_{n+1})|e^{-At}.$$

Hence we obtain

$$\begin{aligned}\mathbb{E}[\|\eta_t-\xi_t\|] &\le \sum_{k=0}^{n-1}\mathbb{E}[\|\xi_t^k-\xi_t^{k+1}\|]+\mathbb{E}[\|\xi_t^n-\eta_t\|] \\ &\le e^{-At}\sum_{k=0}^{n-1}|\eta_0(x_{k+1})-\xi_0(x_{k+1})| + \mathbb{E}[\|\xi_t^n-\eta_t\|] \le e^{-At}\|\eta_0-\xi_0\|+\mathbb{E}[\|\xi_t^n-\eta_t\|].\end{aligned}$$

Since the constants $\sup_{R>0}(C_1(R)+C_5(R))<\infty$, Theorem 4.1 implies that

$$\mathbb{E}[\|\xi_t^n-\eta_t\|]\le\|\xi_0^n-\eta_0\|e^{ct} = \sum_{k=n+1}^{\infty}\eta_0(x_k)e^{ct}\longrightarrow 0,\ \ n\to\infty,$$

where the constant c is independent of n. Hence we obtain $\mathbb{E}[\|\eta_t-\xi_t\|]\le e^{-At}\|\eta_0-\xi_0\|$ which readily yields (6). Since $(\mathcal{X},d)$ is a Polish space, $(\mathcal{P}_1(\mathcal{X}),W_1)$ is a Polish space as well (see e.g. [34, Theorem 6.18]). Therefore the existence and uniqueness of an invariant measure as well as (7) are immediate consequences of (6). This completes the proof. □

8. Linear speed of spread and growth bound

In this section, we prove Theorem 1.3. We consider V to be the vertex set of an infinite connected graph $G=(V,E)$ of bounded degree. Let $\mathrm{dist}(z,z')$ be the graph distance for $z,z'\in V$, and for $x\in V$ and $r>0$ we define

$$\mathbb{B}(x,r) := \{z\in V : \mathrm{dist}(z,x)\le r\}$$

as the set of nodes in V that are within the graph distance r from x. Denote by d the maximum degree of G, that is, the maximum degree of its vertices. Note that $d \geq 2$ since G is connected. For a given $x \in V$ and $k \in \mathbb{N}$ there are at most d^k distinct nodes $y \in V$ satisfying $\mathrm{dist}(x,y) = k$. Hence

$$\#\mathbb{B}(x,r) \leq 1 + d + ... + d^r \leq d^{r+1}, \quad r \in \mathbb{N}. \tag{38}$$

The proof of Theorem 1.3 relies on a heat kernel estimate that is a direct consequence of [7, Corollary 12] (see also [30]) as formulated below.

Lemma 8.1. *Let $(S_t, t \geq 0)$ be a nearest neighbour continuous-time random walk on an infinite connected graph $\widetilde{G}$ of bounded degree with vertex set $\widetilde{V}$. The jump rate from $u \in \widetilde{V}$ to $v \in \widetilde{V}$ is given by $\beta(u,v) > 0$ if $u \sim v$, and 0 otherwise. Here $u \sim v$ indicates that u, v are neighbours. Assume that $\sup_{u \sim v} \beta(u,v) < \infty$ and there exists $m > 0$ such that*

$$\sum_{v:\ v \sim u} \beta(u,v) \geq m > 0, \qquad \forall u \in \widetilde{V}.$$

Let $K(t,u,v) = \mathbb{P}_u\{S_t = v\}$ be the transition probability starting from u to be at v at time t. Then for $u, v \in \widetilde{V}$ and $t \geq 0$

$$K(t,u,v) \leq \frac{1}{m} \exp\Big[-\widetilde{d}(u,v) \ln\Big(\frac{2\widetilde{d}(u,v)}{et} \Big)\Big], \tag{39}$$

where $\widetilde{d}$ is the graph distance in $\widetilde{G}$.

We note that K is also referred to as the heat kernel. To see that Lemma 8.1 does indeed follow from [7, Corollary 12] we take in notation of [7] $b(g) = \beta(g)$, $g \in \widetilde{\mathrm{S}}$, $a(u) = \sum_{v:\ v \sim u} \beta(u,v)$, so that $k \equiv 1$. The constant Λ is defined in [7] as an infimum of the spectrum of a certain operator which in our case can be seen as the generator of $(S_t, t \geq 0)$. The inequalities $0 \leq \Lambda \leq d - 1$ for a d-regular graph follow from [7, Lemma 2], which enables us to drop Λ in (39).

Under the assumptions of Theorem 1.3, recall that $\widetilde{B}$ denotes the effective drift defined in (4). It is convenient to separate the martingale components from the drift of the process. To this end, we rewrite (2) in such a way that all stochastic integrals become martingales, i.e.

$$\begin{aligned} \eta_t(x) = \eta_0(x) &+ \int_0^t \widetilde{B}(x,\eta_s)ds + \int_0^t \sqrt{2c(x,\eta_s(x))}dW_s(x) \\ &+ \sum_{y \in V} \int_0^t \int_{\mathcal{X}} \int_{\mathbb{R}_+} \nu(x) \mathbb{1}_{\{u \leq g(y,\eta_{s-}(y))\}} \widetilde{N}_y(ds,d\nu,du). \end{aligned} \tag{40}$$

where the noise terms are the same as in (N1) – (N4).

Lemma 8.2. *Suppose that the conditions of Theorem 1.3 are satisfied. Then*

$$\int_{\mathcal{X}\setminus\{0\}} \nu(x) H_1(y,d\nu) = 0$$

holds for all $x, y \in V$ such that $\mathrm{dist}(x,y) > R$ and $g(y,\cdot) \neq 0$.

Proof. Using the particular form of $\widetilde{B}$ combined with (8), we obtain for each $x \in V$

$$\sum_{y \in V\setminus\{x\}} g(y,\eta(y)) \int_{\mathcal{X}\setminus\{0\}} \nu(x) H_1(y,d\nu) - B_1(x,\eta(x)) \leq \widetilde{B}(x,\eta) \leq \sum_{y \in V} b(x,y)\eta(y).$$

Take $y \neq x$ arbitrary and $\eta(w) = \varepsilon \mathbb{1}_{\{w=y\}}$, then

$$g(y,\varepsilon) \int_{\mathcal{X}\setminus\{0\}} \nu(x) H_1(y,d\nu) \leq b(x,y)\varepsilon, \qquad \varepsilon > 0,\ x, y \in V,\ x \neq y.$$

Now let $x, y \in V$ be such that $\mathrm{dist}(x,y) > R$ and $g(y,\cdot) \neq 0$. Then $b(x,y) = 0$ and hence $g(y,\varepsilon) \int_{\mathcal{X}\setminus\{0\}} \nu(x) H_1(y,d\nu) = 0$. Since $g(y,\cdot) \neq 0$, we find $\varepsilon > 0$ such that $g(y,\varepsilon) > 0$, which gives $\int_{\mathcal{X}\setminus\{0\}} \nu(x) H_1(y,d\nu) = 0$. □

We are now prepared to prove the result.

Proof of Theorem 1.3. *Step 1.* As a first step, we use a comparison principle to reduce the problem to the case of a constant drift. Let ξ be the unique strong solution of

$$\xi_t(x) = \eta_0(x) + \sum_{y\in V}\int_0^t b(x,y)\xi_s(y)ds + \int_0^t \sqrt{2c(x,\xi_s(x))}dW_s(x)$$
$$+\sum_{y\in V}\int_0^t\int_{\mathcal{X}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\le g(y,\xi_{s-}(y))\}}\widetilde{N}_y(ds,d\nu,du),\quad x\in V.$$

Since $\sum_{y\in V} b(x,y)\eta(y)\ge \widetilde{B}(x,\eta)$ holds for all $x\in V$ and $\eta\in\mathcal{X}$, the comparison principle implies that $\eta_t(x)\le\xi_t(x)$ holds a.s.. Hence, it suffices to prove the assertion for ξ_t. Similarly, letting

$$M := \sup_{x,y\in V} b(x,y) < \infty,$$

we may consider another process $(\zeta_t)_{t\ge 0}$ defined as the unique strong solution of

$$\zeta_t(x) = \eta_0(x) + M\sum_{y\in V}\int_0^t \mathbb{1}_{\{y:\text{dist}(x,y)\le R\}}\zeta_s(y)ds + \int_0^t \sqrt{2c(x,\zeta_s(x))}dW_s(x)$$
$$+\sum_{y\in V}\int_0^t\int_{\mathcal{X}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\le g(y,\zeta_{s-}(y))\}}\widetilde{N}_y(ds,d\nu,du),\quad x\in V.$$

Since $b(x,y)\le M\mathbb{1}_{\{\text{dist}(x,y)\le R\}}$, the comparison principle in Theorem 5.3 yields a.s. $\zeta_t(x)\ge\xi_t(x)$ for all $t\ge 0$ and $x\in V$. Hence, it suffices to prove the assertion for ζ_t.

Step 2. In this step, we derive an estimate of the growth of $\mathbb{E}[\zeta_t(x)]$ concerning $x\in V$ and show (10). For this purpose, consider a new graph $\widehat{G}=(V,\widehat{E})$ with vertex set V, and $u,v\in V$ are neighbours in $\widehat{E}$ if and only if $\text{dist}(x,y)\le R$. Let $\widehat{d}$ be the graph distance on $\widehat{G}$. Then note that $\widehat{d}(x,y)\le\text{dist}(x,y)\le R\widehat{d}(x,y)$ holds for all $x,y\in V$. Consider a continuous-time random walk $(S_t,t\ge 0)$ on $\widehat{G}$ with transition rates $q_{x,y}=M\mathbb{1}_{\{\text{dist}(x,y)\le R\}}$ for $x,y\in V$, and let $K(t,x_0,y)$ be the transition probabilities at time t from x_0 to y. Applying Lemma 8.1 to this random walk on $\widehat{G}$ and using the equivalence of the graph distances $\widehat{d}$ and dist, we find that

$$K(t,x_0,y)\le\frac{1}{M}\exp\Big[-\widehat{d}(x_0,y)\ln\Big(\frac{2\widehat{d}(x_0,y)}{et}\Big)\Big]\le\frac{1}{M}\exp\Big[-\frac{\text{dist}(x_0,y)}{R}\ln\Big(\frac{2\text{dist}(x_0,y)}{eRt}\Big)\Big].$$

In order to relate this bound to the original process ζ_t, let us define $f_x:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ with $x\in V$ as the expectation of $\zeta_t(x)$, i.e., $f_x(t)=\mathbb{E}\zeta_t(x)$. Taking expectation in (40), we get the representation

$$f_t(x) = f_0(x) + M\sum_{y\in V:\ \text{dist}(x,y)\le R}\int_0^t f_s(y)ds.$$

Now, we focus on obtaining an upper bound on $f_x(t)$. The transition probabilities $K(t,x_0,x)$ of $(S_t,t\ge 0)$ satisfy the equation

$$\begin{cases}\frac{d}{dt}K(t,x_0,x) &= -Md_R(x)K(t,x_0,x)+M\sum\limits_{y\in E:1\le\text{dist}(x,y)\le R}K(t,x_0,y),\\ K(t,x_0,x) &= \mathbb{1}_{\{x=x_0\}}.\end{cases}$$

where $d_R(x)=\#\{y\in V:1\le\text{dist}(x,y)\le R\}=\#\mathbb{B}(x,r)-1$ is the number of vertices within distance R from x. Hence $K(t,x_0,\cdot)=e^{tA}\mathbb{1}_{\{x_0\}}$, where $\mathbb{1}_{\{x_0\}}$ is the indicator function and A is a bounded operator on $L^\infty(V)$ defined by

$$Ah(x) = -Md_R(x)h(x) + M\sum_{y\in V:1\le\text{dist}(x,y)\le R}h(y),\qquad h\in L^\infty(V).$$

Consider another bounded linear operator B on $L^\infty(V)$ defined by

$$Bh(x) = -MDh(x) + M \sum_{y\in V:1\leq \mathrm{dist}(x,y)\leq R} h(y), \qquad h \in L^\infty(V),$$

where D is the maximum degree of $\widehat{G}$. Note that by (38), $D \leq d^{R+1} < \infty$, and that $f_\cdot(t) = e^{MDt}e^{tB}\mathbb{1}_{\{x_0\}}$. Finally, (38) implies $d_R(x) \leq D$ for $x \in V$, whence we have $A \geq B$ and hence also $e^{tA} \geq e^{tB}$. This readily yields

$$K(t,x_0,\cdot) = e^{tA}\mathbb{1}_{\{x_0\}} \geq e^{tB}\mathbb{1}_{\{x_0\}} = e^{-MDt}f_\cdot(t), \quad t \geq 0,$$

and hence $K(t,x_0,y) \geq f_y(t)e^{-MDt}$ for all $y \in V$. In view of (12), we find constants $C_v, \ell > 0$ such that

$$v(x) \geq C_v D^{-\ell \mathrm{dist}(x_0,x)}, \qquad x \in V. \tag{41}$$

Define the constant

$$C_0 = \max\left\{1, \ \frac{MDR}{\ln(D)}, \ \frac{D^{(2\ell+5)R}eR}{2}\right\}.$$

Using the above estimates we obtain for all $y \in V$ and $t > 0$ satisfying $\mathrm{dist}(x_0,y) > C_0 t$ that

$$\begin{aligned} f_y(t) &\leq \frac{e^{MDt}}{M}\exp\left[-\frac{\mathrm{dist}(x_0,y)}{R}\ln\left(\frac{2\mathrm{dist}(x_0,y)}{eRt}\right)\right] \\ &\leq \frac{1}{M}\exp\left[\left(\frac{MD}{C_0} - \frac{1}{R}\ln\left(\frac{2C_0}{eR}\right)\right)\mathrm{dist}(x_0,y)\right] \\ &\leq \frac{1}{M}\exp\left[\left(\frac{MD}{C_0} - \frac{(2\ell+5)\ln(D)}{R}\right)\mathrm{dist}(x_0,y)\right] \leq \frac{D^{-2(\ell+2)\mathrm{dist}(x_0,y)/R}}{M}. \end{aligned} \tag{42}$$

Step 3. Next, we prove a similar estimate for the expected supremum of the process, i.e., for $\mathbb{E}\left[\sup_{s\in[0,t]}\zeta_s(x)\right]$. Let $x \neq x_0$ and $t > 0$. Recall that $\eta_0(x) = 0$. Then, by (40), we arrive at

$$\begin{aligned} \sup_{r\in[0,t]}\zeta_r(x) \leq M &\sum_{y\in V:\ \mathrm{dist}(x,y)\leq R}\int_0^t \zeta_s(y)ds + \sup_{r\in[0,t]}\left|\int_0^r \sqrt{2c(x,\zeta_s(x))}dW_s(x)\right| \\ &+ \sum_{y\in V\setminus\{x\}}\sup_{r\in[0,t]}\left|\int_0^r\int_{\mathcal{X}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\leq g(y,\zeta_{s-}(y))\}}\widetilde{N}_y(ds,d\nu,du)\right|. \\ &+ \sup_{r\in[0,t]}\left|\int_0^r\int_{\mathcal{X}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\leq g(x,\zeta_{s-}(x))\}}\widetilde{N}_x(ds,d\nu,du)\right|. \end{aligned}$$

Let us bound all terms in expectation. Doob's maximal inequality applied to the continuous martingale gives by (A2)

$$\begin{aligned} \mathbb{E}\left[\sup_{r\in[0,t]}\left|\int_0^r\sqrt{c(x,\zeta_s(x))}dW_s(x)\right|^2\right] &\leq 4C_2(x)\int_0^t f_s(x)ds \\ &\leq 4\left(\sum_{y\in V}v(y)C_2(y)\right)\frac{1}{v(x)}\int_0^t f_s(x)ds \end{aligned}$$

For the sum against $\widetilde{N}_y$ with $x \neq y$ we obtain from (A3)

$$\begin{aligned} &\sum_{y\in V\setminus\{x\}}\mathbb{E}\left[\sup_{r\in[0,t]}\left|\int_0^r\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+}\nu(x)\mathbb{1}_{\{u\leq g(y,\zeta_{s-}(y))\}}\widetilde{N}_y(ds,d\nu,du)\right|\right] \\ &\leq 2\sum_{y\in V\setminus\{x\}}\int_0^t\int_{\mathcal{X}\setminus\{0\}}\nu(x)\mathbb{E}[g(y,\zeta_{s-}(y))]H_1(y,d\nu)ds \end{aligned}$$

$$\leq 2 \sum_{y \in V \setminus \{x\}} C_3(y) \int_{\mathcal{X} \setminus \{0\}} \nu(x) H_1(y, d\nu) \int_0^t f_y(s) ds.$$

Finally, for the integrals against $\widetilde{N}_x$ we consider $\{\|\nu\| \leq 1\} \setminus \{0\}$ and $\{\|\nu\| > 1\}$ separately. Namely, we obtain from the Burkholder-Davis-Gundy inequality for Poisson random measures and then (A3) combined with (A4)

$$\begin{aligned}
&\mathbb{E}\left[\sup_{r \in [0,t]} \left| \int_0^r \int_{\{\|\nu\| \leq 1\} \setminus \{0\}} \int_{\mathbb{R}_+} \nu(x) \mathbb{1}_{\{u \leq g(x, \zeta_{s-}(x))\}} \widetilde{N}_x(ds, d\nu, du) \right|^2 \right] \\
&\leq 4 \int_0^t \int_{\{\|\nu\| \leq 1\} \setminus \{0\}} \nu(x)^2 g(x, \zeta_s(x)) H_1(x, d\nu) ds \\
&\leq 4 C_3(x) \left(\int_{\{\|\nu\| \leq 1\} \setminus \{0\}} \nu(x)^2 H_1(x, d\nu) \right) \int_0^t f_s(x) ds \\
&\leq 4 \left(\sum_{y \in V} C_3(y) \left(\int_{\{\|\nu\| \leq 1\} \setminus \{0\}} \nu(y)^2 H_1(y, d\nu) \right) \right) \frac{1}{v(x)} \int_0^t f_s(x) ds,
\end{aligned}$$

while for the big jumps, we obtain from (A4)

$$\begin{aligned}
&\mathbb{E}\left[\sup_{r \in [0,t]} \left| \int_0^r \int_{\{\|\nu\| > 1\}} \int_{\mathbb{R}_+} \nu(x) \mathbb{1}_{\{u \leq g(x, \zeta_{s-}(x))\}} \widetilde{N}_x(ds, d\nu, du) \right| \right] \\
&\leq 2 \mathbb{E}\left[\int_0^t \int_{\{\|\nu\| > 1\}} \int_{\mathbb{R}_+} \nu(x) \mathbb{1}_{\{u \leq g(x, \zeta_{s-}(x))\}} H_1(x, d\nu) du ds \right] \\
&= 2 \left(\int_{\{\|\nu\| > 1\}} \nu(x) H_1(x, d\nu) \right) \int_0^t \mathbb{E}[g(x, \zeta_s(y))] ds \\
&\leq 2 C_3(x) \left(\int_{\{\|\nu\| > 1\}} \nu(x) H_1(x, d\nu) \right) \int_0^t f_s(x) ds \leq 2 C_4 \int_0^t f_s(x) ds.
\end{aligned}$$

Combining all these estimates, we find a constant $C > 0$ such that

$$\begin{aligned}
\mathbb{E}\left[\sup_{r \in [0,t]} \zeta_r(x) \right] \leq M &\sum_{y \in V:\ \mathrm{dist}(x,y) \leq R} \int_0^t f_s(y) ds \\
&+ 2 \sum_{y \in V \setminus \{x\}} C_3(y) \left(\int_{\mathcal{X} \setminus \{0\}} \nu(x) H_1(y, d\nu) \right) \int_0^t f_s(y) ds \\
&+ \frac{C}{\sqrt{v(x)}} \left(\int_0^t f_s(x) ds \right)^{1/2} + 2 C_4 \int_0^t f_s(x) ds
\end{aligned}$$

holds for each $x \in V$. Let $t_0 > 0$ be arbitrary. Letting $t > t_0$ and $x \in V$ be such that

$$\mathrm{dist}(x_0, x) > \left(C_0 + \frac{R}{t_0} \right) t,$$

we find for $y \in V$ satisfying $\mathrm{dist}(x, y) \leq R$ that

$$d(x_0, y) \geq d(x, x_0) - d(x, y) \geq \left(C_0 + \frac{R}{t_0} \right) t - R > C_0 t.$$

Hence we can use the previously shown inequality (42) on $f_y(s)$ for $s \in (0,t]$ from Step 2 to find that

$$\begin{aligned}\mathbb{E}\left[\sup_{r\in[0,t]} \zeta_r(x)\right] \leq &\sum_{y\in V:\ \mathrm{dist}(x,y)\leq R} tD^{-2(\ell+2)\mathrm{dist}(x_0,y)/R} \\ &+ \frac{2}{M}\sum_{y\in V\setminus\{x\}} C_3(y)\left(\int_{\mathcal{X}\setminus\{0\}} \nu(x)H_1(y,d\nu)\right) tD^{-2(\ell+2)\mathrm{dist}(x_0,y)/R} \\ &+ \frac{C}{\sqrt{M}}\sqrt{\frac{t}{v(x)}}D^{-(\ell+2)\mathrm{dist}(x_0,x)/R} + \frac{2C_4}{M}tD^{-2(\ell+2)\mathrm{dist}(x_0,x)/R}.\end{aligned}$$

Since the graph $G=(V,E)$ is connected, the maximum degree satisfies $D>1$. For the first and second terms, we use the elementary inequality $xD^{-x} \leq \frac{1}{e\ln(D)}$, $x>0$, to find

$$\begin{aligned}tD^{-2(\ell+2)\mathrm{dist}(x_0,y)/R} &< \frac{\mathrm{dist}(x_0,y)R^{-1}D^{-\mathrm{dist}(x_0,y)/R}}{C_0}RD^{-(2\ell+3)\mathrm{dist}(x_0,y)/R} \\ &\leq \frac{R}{e\ln(D)C_0}D^{-\mathrm{dist}(x_0,y)/R} \\ &\leq \frac{RD^R}{e\ln(D)C_0}D^{-\mathrm{dist}(x_0,x)/R}.\end{aligned}$$

Similarly, we obtain for the last term

$$tD^{-2(\ell+2)\mathrm{dist}(x_0,x)/R} \leq \frac{RD^R}{e\ln(D)C_0}D^{-\mathrm{dist}(x_0,x)/R}.$$

For the remaining third term, we use (41) and the elementary inequality $\sqrt{x}D^{-x} \leq \frac{1}{\sqrt{2e\ln(D)}}$ to find that

$$\begin{aligned}\sqrt{\frac{t}{v(x)}}D^{-(\ell+2)\mathrm{dist}(x_0,x)/R} &\leq C_v\frac{\sqrt{R^{-1}\mathrm{dist}(x_0,x)}D^{-\mathrm{dist}(x_0,x)/R}}{\left(C_0+\frac{R}{t_0}\right)^{1/2}}R^{1/2}D^{-\mathrm{dist}(x_0,x)/R} \\ &\leq \frac{R^{1/2}D^{-\mathrm{dist}(x_0,x)/R}}{\sqrt{2e\ln(D)\left(C_0+\frac{R}{t_0}\right)}}.\end{aligned}$$

Thus, combining these estimates gives for $t>t_0$ and some constant $C'>0$ independent of x,y,t the estimate

$$\begin{aligned}\mathbb{E}\left[\sup_{r\in[0,t]} \zeta_r(x)\right] &\leq C'D^{-\mathrm{dist}(x_0,x)/R} \\ &\quad + C'\sum_{y\in V\setminus\{x\}} C_3(y)\int_{\mathcal{X}\setminus\{0\}} \nu(x)H_1(y,d\nu)D^{-\mathrm{dist}(x_0,x)/R} \\ &\leq \left(C' + C_4 d^{R+1}\sup_{\mathrm{dist}(x,y)\leq R}\frac{v(y)}{v(x)}\right)D^{-\mathrm{dist}(x_0,x)/R}\end{aligned}$$

where we have used Lemma 8.2 to find

$$\begin{aligned}&\sum_{y\in V\setminus\{x\}} C_3(y)\int_{\mathcal{X}\setminus\{0\}} \nu(x)H_1(y,d\nu) \\ &\leq \sum_{y:\ \mathrm{dist}(y,x)\leq R,\ y\neq x} \mathbb{1}_{\{C_3(y)>0\}}\frac{C_3(y)}{v(x)}\int_{\mathcal{X}\setminus\{0\}}\sum_{w\in V\setminus\{y\}} v(w)\nu(w)H_1(y,d\nu)\end{aligned}$$

$$\leq C_4 \sum_{y:\ \mathrm{dist}(y,x)\leq R,\ y\neq x} \frac{v(y)}{v(x)}$$

$$\leq C_4 d^{R+1} \sup_{\mathrm{dist}(x,y)\leq R} \frac{v(y)}{v(x)} < \infty.$$

and have set, without loss of generality, $C_3(y) = 0$ whenever $g(y,\cdot) = 0$.

Step 4. In this last step, we derive the assertion from the Borel-Cantelli lemma. Namely, letting $C_1 = \left(C_0 + \frac{R}{t_0}\right)$, we obtain for $t \geq t_0$ and $\varepsilon > 0$, the estimate

$$\sum_{t\in\mathbb{N}} \mathbb{P}\left[\sup_{x\in V:|x|>C_1 t} \sup_{r\in[0,t]} \zeta_r(x) > \varepsilon\right] \leq \frac{1}{\varepsilon}\sum_{t\in\mathbb{N}} \sum_{\substack{x\in V:\\ |x|>C_1 t}} \mathbb{E}\left[\sup_{r\in[0,t]} \zeta_r(x)\right]$$

$$\leq \frac{C''}{\varepsilon}\sum_{t\in\mathbb{N}} \sum_{\substack{x\in V:\\ |x|>C_1 t}} D^{-\mathrm{dist}(x_0,x)/R}.$$

Since $D > 1$, the right-hand side is finite, and we may apply the Borel-Cantelli lemma. This gives

$$\mathbb{P}\left[\sup_{x\in V:|x|>C_1 t} \sup_{r\in[0,t]} \zeta_r(x) > \varepsilon \text{ for infinitely many } t\in\mathbb{N}\right] = 0$$

which concludes the proof. □

Appendix A: Transformation of state-space

Let $v \neq 1$ be some weight function and denote by $\mathcal{X}_v$ the corresponding state space given by (1). Furthermore, let $\mathcal{X}_1$ be given by (1) with the particular choice $v \equiv 1$. Let us define

$$T_v : \mathcal{X}_1 \longrightarrow \mathcal{X}_v, \qquad T_v\eta(x) = v(x)^{-1}\eta(x),\ x\in V.$$

Then T_v is continuous and bijective with continuous inverse is given by $T_v^{-1} = T_{v^{-1}}$. This transformation simplifies the proofs by considering only the special case where $v \equiv 1$. The precise statement is given below.

Lemma A.1. *A process $(\eta_t)_{t\geq 0}$ is a weak solution of (2) on $\mathcal{X}_1$ with coefficients (B,c,g,ρ,H_1,H_2) if and only if $\widetilde{\eta}_t = T_v\eta_t$ is a weak solution of (2) on $\mathcal{X}_v$ with coefficients $(\widetilde{B},\widetilde{c},\widetilde{g},\widetilde{\rho},\widetilde{H}_1,\widetilde{H}_2)$ given by $\widetilde{B}(x,\eta) = v(x)^{-1}B(x,T_{v^{-1}}\eta)$, $\widetilde{c}(x,t) = v(x)^{-2}c(x,tv(x))$, $\widetilde{g}(x,t) = g(x,tv(x))$, $\widetilde{\rho}(x,\eta,\nu) = \rho(x,T_{v^{-1}}\eta,T_{v^{-1}}\nu)$, $\widetilde{H}_1(y,d\nu) = H_1(y,d\nu)\circ T_v^{-1}$, and $\widetilde{H}_2(d\nu) = H_2(d\nu)\circ T_v^{-1}$.*

Moreover, $(\widetilde{B},\widetilde{c},\widetilde{g},\widetilde{\rho},\widetilde{H}_1,\widetilde{H}_2)$ satisfy conditions (A1) – (A6) with the given weight function v if and only if (B,c,g,ρ,H_1,H_2) satisfy (A1) – (A6) with $v\equiv 1$.

Proof. Suppose that $(\eta_t)_{t\geq 0}$ is a weak solution of (2) on $\mathcal{X}_1$ with coefficients (B,c,g,ρ,H_1,H_2). By multiplication of (2) with $v(x)^{-1}$, it is easy to see that $\widetilde{\eta}_t = T_v\eta_t$ is a weak solution of (2) with the adjusted coefficients $\widetilde{B},\widetilde{c},\widetilde{g},\widetilde{\rho}$, and new Poisson random measures $\mathcal{N}_y := N_y \circ (\mathrm{id}_{\mathbb{R}_+}, T_v, \mathrm{id}_{\mathbb{R}_+})^{-1}$ for $y\in V$, and $\mathcal{M} := M \circ (\mathrm{id}_{\mathbb{R}_+}, T_v, \mathrm{id}_{\mathbb{R}_+})^{-1}$. Since these transformed Poisson random measures have compensators $\widehat{\mathcal{N}}_y(ds,d\nu,du) = ds\widetilde{H}_1(y,d\nu)du$ and $\widehat{\mathcal{M}}(ds,d\nu,du) = ds\widetilde{H}_2(d\nu)du$, it follows that $\widetilde{\eta}$ is a weak solution of (2) with the desired coefficients. Conversely, if $\widetilde{\eta}_t$ is a weak solution of (2) on $\mathcal{X}_v$ with $(\widetilde{B},\widetilde{c},\widetilde{g},\widetilde{\rho},\widetilde{H}_1,\widetilde{H}_2)$, then a similar computation shows that $\eta_t = T_{v^{-1}}\widetilde{\eta}_t$ is a weak solution of (2) on $\mathcal{X}_1$ with (B,c,g,ρ,H_1,H_2). The equivalence of conditions (A1) – (A6) is left to the reader. □

Appendix B: Martingale property for (24)

In this section, we will show that $(\mathcal{M}(t\wedge\tau_m))_{t\geq 0}$ is a martingale for each $m,k\geq 1$. Firstly, recall that we have set $v\equiv 1$. Then let us write $\mathcal{M}(t\wedge\tau_m) = \sum_{j=1}^5 \mathcal{M}_j(t\wedge\tau_m)$ with

$$\mathcal{M}_1(t\wedge\tau_m) := \int_0^{t\wedge\tau_m} \phi_k'(\zeta_s(x))\left(\sqrt{2c(x,\eta_s(x))} - \sqrt{2c(x,\xi_s(x))}\right) dW_s(x),$$

$$\mathcal{M}_2(t\wedge\tau_m) := \int_0^{t\wedge\tau_m}\int_{\{\|\nu\|\le 1\}\setminus\{0\}}\int_{\mathbb{R}_+} D_{\Delta_0(x,s)}\phi_k(\zeta_{s-}(x))\widetilde{N}_x(ds,d\nu,du),$$

$$\mathcal{M}_3(t\wedge\tau_m) := \int_0^{t\wedge\tau_m}\int_{\{\|\nu\|>1\}}\int_{\mathbb{R}_+} D_{\Delta_0(x,s)}\phi_k(\zeta_{s-}(x))\widetilde{N}_x(ds,d\nu,du),$$

$$\mathcal{M}_4(t\wedge\tau_m) := \sum_{y\in V\setminus\{x\}}\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+} D_{\Delta_0(y,s)}\phi_k(\zeta_{s-}(x))\widetilde{N}_y(ds,d\nu,du),$$

$$\mathcal{M}_5(t\wedge\tau_m) := \int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+} D_{\Delta_1(x,s)}\phi_k(\zeta_{s-}(x))\widetilde{M}(ds,d\nu,du).$$

Then, it suffices to prove the following lemma:

Lemma B.1. *Under the notation of Section 2, the following holds:*

- *(a)* $(\mathcal{M}_1(t\wedge\tau_m))_{t\ge 0}$ *is a continuous square-integrable martingale;*
- *(b)* $(\mathcal{M}_2(t\wedge\tau_m))_{t\ge 0}$ *is a square-integrable martingale;*
- *(c)* $(\mathcal{M}_3(t\wedge\tau_m))_{t\ge 0}$ *is an integrable martingale;*
- *(d)* $(\mathcal{M}_4(t\wedge\tau_m))_{t\ge 0}$ *is an integrable martingale.*
- *(e)* $(\mathcal{M}_5(t\wedge\tau_m))_{t\ge 0}$ *is an integrable martingale.*

Proof. (a) Using first Ito's isometry, then $|\phi_k'|\le 1$ and $(a-b)^2\le|a^2-b^2|$ for $a,b\ge 0$, we obtain

$$\begin{aligned}\mathbb{E}\left[|\mathcal{M}_1(t\wedge\tau_m)|^2\right] &= \mathbb{E}\left[\int_0^{t\wedge\tau_m}\phi_k'(\zeta_s(x))^2\left(\sqrt{2c(x,\eta_s(x))}-\sqrt{2c(x,\xi_s(x))}\right)^2 ds\right]\\ &\le 2\mathbb{E}\left[\int_0^{t\wedge\tau_m}|c(x,\eta_{s-}(x))-c(x,\xi_{s-}(x))|\,ds\right]\\ &\le 2C_2(x)\mathbb{E}\left[\int_0^{t\wedge\tau_m}|\eta_{s-}(x)-\xi_{s-}(x)|ds\right]\le 4mtC_2(x)<\infty,\end{aligned}$$

where we have used (25) and (A2).

(b) Recall that $\widehat{N}_x(ds,d\nu,du)=dsH_1(x,d\nu)du$, so that the assertion follows from

$$\begin{aligned}&\mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\{\|\nu\|\le 1\}\setminus\{0\}}\int_{\mathbb{R}_+}|D_{\Delta_0(x,s)}\phi_k(\zeta_{s-}(x))|^2 dsH_1(x,d\nu)du\right]\\ &\le \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\{\|\nu\|\le 1\}\setminus\{0\}}\int_{\mathbb{R}_+}|\Delta_0(x,s)|^2 dsH_1(x,d\nu)du\right]\\ &= \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\{\|\nu\|\le 1\}\setminus\{0\}}\nu(x)^2|g(x,\eta_{s-}(x))-g(x,\xi_{s-}(x))|dsH_1(x,d\nu)\right]\\ &\le C_3(x)\left(\int_{\{\|\nu\|\le 1\}\setminus\{0\}}\nu(x)^2H_1(x,d\nu)\right)\mathbb{E}\left[\int_0^{t\wedge\tau_m}|\eta_{s-}(x)-\xi_{s-}(x)|ds\right]\\ &\le 2C_3(x)\left(\int_{\{\|\nu\|\le 1\}\setminus\{0\}}\nu(x)^2H_1(x,d\nu)\right)mt<\infty,\end{aligned}$$

where we have used (25) and (A3), (A4).

(c) Analogously to the estimates in part (b), we obtain

$$\begin{aligned}&\mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\{\|\nu\|>1\}}\int_{\mathbb{R}_+}\left|D_{\Delta_0(x,s)}\phi_k(\zeta_{s-}(x))\right| dsH_1(x,d\nu)du\right]\\ &\le \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\{\|\nu\|>1\}}\int_{\mathbb{R}_+}|\Delta_0(x,s)|\,dsH_1(x,d\nu)du\right]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\{\|\nu\|>1\}} \nu(x)\,|g(x,\eta_{s-}(x)) - g(x,\xi_{s-}(x))|\,dsH_1(x,d\nu)\right] \\
&\le C_3(x)\left(\int_{\{\|\nu\|>1\}} \nu(x)H_1(x,d\nu)\right)\mathbb{E}\left[\int_0^{t\wedge\tau_m} |\eta_{s-}(x)-\xi_{s-}(x)|ds\right] \\
&\le 2C_3(x)\left(\int_{\{\|\nu\|>1\}} \nu(x)H_1(x,d\nu)\right)mt < \infty.
\end{aligned}
$$

(d) Similarly, we obtain in this case

$$
\begin{aligned}
\mathbb{E}\left[|\mathcal{M}_4(t\wedge\tau_m)|\right] &\le \sum_{y\in V\setminus\{x\}} \mathbb{E}\left[\left|\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+} D_{\Delta_0(y,s)}\phi_k(\zeta_{s-}(x))\widetilde{N}_y(ds,d\nu,du)\right|\right] \\
&\le 2\sum_{y\in V\setminus\{x\}} \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+} \left|D_{\Delta_0(y,s)}\phi_k(\zeta_{s-}(x))\right| dsH_1(y,d\nu)du\right] \\
&\le 2\sum_{y\in V\setminus\{x\}} C_3(y)\left(\int_{\mathcal{X}\setminus\{0\}} \nu(x)H_1(y,d\nu)\right)\mathbb{E}\left[\int_0^{t\wedge\tau_m} |\eta_{s-}(y)-\xi_{s-}(y)|ds\right] \\
&\le 2\sum_{z\in V}\sum_{y\in V\setminus\{z\}} C_3(y)\left(\int_{\mathcal{X}\setminus\{0\}} \nu(z)H_1(y,d\nu)\right)\mathbb{E}\left[\int_0^{t\wedge\tau_m} |\eta_{s-}(y)-\xi_{s-}(y)|ds\right] \\
&= 2\sum_{y\in V} C_3(y)\left(\int_{\mathcal{X}\setminus\{0\}}\sum_{z\in V\setminus\{y\}} \nu(z)H_1(y,d\nu)\right)\mathbb{E}\left[\int_0^{t\wedge\tau_m} |\eta_{s-}(y)-\xi_{s-}(y)|ds\right] \\
&\le 2C_4\sum_{y\in V}\mathbb{E}\left[\int_0^{t\wedge\tau_m} |\eta_{s-}(y)-\xi_{s-}(y)|ds\right] \le 4mtC_4 < \infty,
\end{aligned}
$$

where we have used (A4). Hence the series is absolutely convergent in L^1 and thus $(\mathcal{M}_4(t\wedge\tau_m))_{t\ge 0}$ is a martingale.

(e) Using that $\widehat{M}(ds,d\nu,du) = dsH_2(d\nu)du$, the assertion follows from

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}}\int_{\mathbb{R}_+} \left|D_{\Delta_1(x,s)}\phi_k(\zeta_{s-}(x))\right| dsH_2(d\nu)du\right] \\
&\le \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}} \nu(x)|\rho(x,\eta_s,\nu)-\rho(x,\xi_s,\nu)|H_2(d\nu)ds\right] \\
&\le \mathbb{E}\left[\int_0^{t\wedge\tau_m}\int_{\mathcal{X}\setminus\{0\}}\sum_{y\in V} \nu(y)|\rho(y,\eta_s,\nu)-\rho(y,\xi_s,\nu)|H_2(d\nu)ds\right] \\
&\le C_5(m)\mathbb{E}\left[\int_0^{t\wedge\tau_m} \|\eta_s-\xi_s\|ds\right] \le 2mC_5(m)t < \infty
\end{aligned}
$$

where we have used (A5). □

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