Potential Analysis
https://doi.org/10.1007/s11118-021-09983-4

Long-Time Behavior for Subcritical Measure-Valued Branching Processes with Immigration

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Received: 13 March 2019 / Accepted: 22 December 2021

Abstract

In this work we study the long-time behavior for subcritical measure-valued branching processes with immigration on the space of tempered measures. Under some reasonable assumptions on the spatial motion, the branching and immigration mechanisms, we prove the existence and uniqueness of an invariant probability measure for the corresponding Markov transition semigroup. Moreover, we show that it converges with exponential rate to the unique invariant measure in the Wasserstein distance as well as in a distance defined in terms of Laplace transforms. Finally, we consider an application of our results to super-Lévy processes as well as branching particle systems on the lattice with noncompact spins.

Keywords Dawson-Watanabe superprocess · Measure-valued Markov process · Branching · Ergodicity · Invariant measure · Immigration

Mathematics Subject Classification (2010) Primary 60J68, 60J80 · Secondary 60B10

1 Introduction

Measure-valued branching processes have been first studied by Watanabe 1968 [44], Silverstein 1969 [40] and Kawazu, Watanabe 1971 [25] where they have been derived as scaling limits of Galton-Watson processes. For a detailed introduction on measure-valued Markov processes (also called superprocesses) we refer to Dynkin [8], Etheridge [11], Perkins [37], Le Gall [31] and Li [33]. A measure-valued branching process with immigration is a Markov process whose Markov transition kernel $P_t(\mu, d\nu)$ has Laplace transform

$$\int_{M(E)} e^{-\langle f,\nu\rangle} P_t(\mu, d\nu) = \exp\left(-\langle V_t f, \mu\rangle - \int_0^t \psi(V_s f)ds\right), \qquad t \geq 0,$$

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where $\langle f, \nu\rangle = \int_E f(x)\nu(dx)$ and $v_t(x) := V_t f(x)$ is the unique nonnegative mild solution to

$$\frac{\partial v_t(x)}{\partial t} = Av_t(x) - \phi(x, v_t), \qquad v_0 = f \in D(A). \tag{1.1}$$

Here E denotes the location space, $(A, D(A))$ the generator of a transition semigroup describing the underlying spatial motion, $M(E)$ the space of all finite Borel measures on E, ϕ the branching and ψ the immigration mechanism.

One of the most prominent examples is the super-Brownian motion where $E = \mathbb{R}^d$, $A = \frac{1}{2}\Delta$, $\psi = 0$ and $\phi(x, f) = \frac{1}{2}f(x)^2$, see, e.g., [29, 38] and [39] where it is shown that in dimension $d = 1$ this process has a density field $(X_t(x))_{x\in\mathbb{R},\, t\geq 0}$ which is the unique weak solution to

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x)}\dot{W}_t(x), \qquad t > 0, \ \ x \in \mathbb{R}, \tag{1.2}$$

where $\dot{W}_t(x)$ is the derivative of a space-time Gaussian white noise. Note that pathwise uniqueness for Eq. 1.2 remains an open problem. However, still in dimension $d = 1$, pathwise uniqueness for the distribution function was recently obtained in [45]. The latter result was extended in [19] to super-Lévy processes with branching mechanism

$$\phi(x, f) = bf(x) + cf(x)^2 + \int_0^\infty \left(e^{-zf(x)} - 1 + zf(x)\right) m(dz), \tag{1.3}$$

where $b \in \mathbb{R}$, $c \geq 0$ and m is a Borel measure on $(0, \infty)$ with $\int_0^\infty u \wedge u^2 m(du) < \infty$. Let us also mention other interesting related results such as [5, 7] and [36]. It is worthwile to mention that solutions to nonlinear Partial Differential Equations of the form Eq. 1.1 can be simulated by a Monte Carlo algorithm based on the associated branching process, see [20] and the references therein.

Another natural and interesting example studied in the literature corresponds to branching particle systems where E is a countable discrete set and A the generator of a Markov chain on E. Indeed, the particular case $E = \mathbb{N}$ with $A = 0$ was recently studied by Kyprianou and Palau [30] where precise conditions for local and global extinction have been obtained. The case $E = \mathbb{Z}^d$ falls into the framework of interacting particle systems on the lattice with noncompact spins as studied by Bezborodov, Kondratiev and Kutoviy [2], see also the references therein. Note that in both cases we may identify

$$M(E) \cong \left\{ (\mu(x))_{x\in E} \in \mathbb{R}_+^E \mid \sum_{x\in E} \mu(x) < \infty \right\},$$

which reveals the underlying lattice structure.

In this work we study the long-time behavior of subcritical measure-valued branching processes including the aforementioned cases of super-Lévy processes as well as branching particle systems on the lattice with noncompact spins. We provide a sufficient condition on the branching and immigration mechanisms such that the associated Markov process admits a unique invariant probability measure π. Afterwards we establish $P_t(\mu, \cdot) \longrightarrow \pi$ for $t \to \infty$ and study the corresponding rate of convergence in two different distances.

Let us briefly mention some known results from the literature. In the particular case $E = \{0\}$ one has $M(\{0\}) \cong \mathbb{R}_+$ and the most general branching mechanism is of the form Eq. 1.3. It describes a one-dimensional *continuous-state branching process with immigration* first introduced in the framework of diffusions by Feller 1951 [14] and then for more general cases by Jiřina 1958 [24]. Its long-time behavior was studied in [26], [33, Chapter 3], [34] and more recently in [15]. More generally, the case $E = \{1, \ldots, d\}$ with $A = 0$

corresponds to *multi-type continuous-state branching processes with immigration* for which the corresponding state space is identified by $M(\{1,\dots,d\}) \cong \mathbb{R}_+^d$, see [1]. In contrast to $E = \{0\}$, here the most general branching mechanism involves the possibility that an individual of type j produces other individuals of types $k \neq j$ which results in a nonlocal branching mechanism. The first general result on invariant measures and ergodicity for multi-type CBI processes was recently obtained in [16, 23] where actually the larger class of affine processes on the canonical state space $\mathbb{R}_+^d \times \mathbb{R}^n$ was investigated, see also [35] and [46] for related results on this topic. For arbitrary compact spaces E, Stannat has studied in [41] and [42] the case of an immigration mechanism $\psi(f) = \langle f, \beta\rangle$ and branching mechanism

$$\phi(x, f) = c(x)f(x)^2 + b(x)f(x) + \int_0^\infty \left(e^{-f(x)z} - 1 + f(x)z\right) m(x, dz), \qquad (1.4)$$

where β is a finite measure on E, $c \geq 0$ and b are continuous and bounded while $m(x, dz)$ is a kernel of positive measures satisfying $\sup_{x\in E} \int_0^\infty z \wedge z^2 m(x, dz) < \infty$. Note that $\phi(x, f)$ given by Eq. 1.4 is local in f and describes at each point $x \in E$ the branching mechanism of a *one-dimensional continuous-state branching process*. Another related result on compact state spaces was obtained in [12].

Contrary, in this work E is a Lusin topological space and ϕ, ψ are general (including nonlocal) branching and immigration mechanisms in the sense of [33, Chapters 2, 6 and 9], see Section 2 here for details. In particular, our results also cover the case of super-Lévy processes with immigration as well as branching particle systems on the lattice as studied in [30]. Contrary to the classical case of measure-valued branching processes supported on the space of finite Borel measures $M(E)$, the results obtained in this work also cover branching Markov processes supported on the space of tempered measures over E. The latter ones can be used to treat population models where the total number of individuals is infinite. Tempered measures have been first introduced for the study of critical branching Markov processes without immigration ($\psi = 0$) where the long-time behavior of the Markov process started at the Lebesgue measure was investigated, see Iscoe [22]. Further developments in this direction have been obtained in [3, 4, 6, 10, 27] and [28]. For branching particle systems ergodic results and properties of the invariant occupation measure have been also studied in [18, 21].

Another motivation to study branching processes on the space of tempered measures is related to possible immigration of mass (or particles). Namely, consider a branching process on $E = \mathbb{R}^d$ and suppose that mass immigrates according to a Poisson point process with intensity $\beta(dx)$. If β is a finite measure (e.g. $\beta(dx) = 1_{\{|x|\leq 1\}}dx$ or $\beta(dx) = \delta_0(dx)$), then only a finite amount of mass immigrates into the system in finite time. However, applications modelled by translation invariant rates such as $\beta(dx) = dx$ share the common feature that β is an infinite measure and hence an infinite amount of mass enters the system in finite time. The temperedness of measures is used to prevent the system to accumulate an infinite amount of mass in some compact set.

Our results are mainly based on two different methods inspired by existing results for affine processes on the canonical state space (see [16, 23]), but now applied to measure-valued Markov processes. Indeed, as shown in Section 4 we prove convergence of the Laplace Transfom and then derive from this existence and uniqueness of an invariant measure (compare with [23]). While the results from Section 4 may also have been obtained by different (probabilistic coupling) methods, our approach based on convergence of the Laplace Transform seems to be more elementary and additionally points out a relation between affine processes on the canonical state space and measure-valued Markov processes. Such a relation is then further exploited in Section 5 where we also provide a convergence rate in the Wasserstein distance. To prove an estimate in the Wasserstein distance one typically needs to find a reasonable coupling for the Markov process, which is, in general, a challenging mathematical task. For affine processes on the canoncial state space a pathwise construction combined with a comparison principle was proposed in [16] in order to effectively estimate the Wasserstein distance by the first moment of the process. While in the measure-valued setting there does not exist an analogous pathwise construction, we will show in Section 5 that it is still possible to obtain the same result solely from the branching property. Such observation has been recently used in [32] to derive also other types of estimates for possibly different classes of processes.

This work is organized as follows. In Section 2 we introduce the class of measure-valued branching processes with immigration studied in this work. Although all results stated in Section 2 should be well-known among experts, we provide, whenever we were not able to find an adequate reference, some additional comments and sketches of proofs. In this way we intend to make this work accessible to a wide audience. In Section 3 we briefly study the long-time behavior of the first moment, i.e., the mean density of the process. Section 4 is devoted to the existence and uniqueness of invariant measures whereas an exponential rate of convergence towards the unique invariant measure is investigated in Section 5. Finally the example of super-Lévy processes and measure-valued particle systems is considered in Section 6. Some auxilliary results are collected in the Appendix A.

2 Measure-Valued Branching Processes with Immigration

Let E be a Lusin topological space (e.g. a Polish space). Fix a strictly positive, continuous function h on E. Let $B_h(E)$ be the Banach space of real-valued Borel functions on E equipped with the norm

$$\|f\|_h := \sup_{x \in E} \frac{|f(x)|}{h(x)} < \infty.$$

Denote by $C_h(E)$ its closed subspace of continuous functions satisfying $\|f\|_h < \infty$ and let $B_h(E)^+, C_h(E)^+$ be the corresponding cones of nonnegative functions. Let $M_h(E)$ be the space of tempered measures over E, i.e.

$$M_h(E) = \left\{ \mu \text{ Borel measure on } E \ : \ \int_E h(x)\mu(dx) < \infty \right\}.$$

For $f \in B_h(E)$ and $\mu \in M_h(E)$ we set $\langle f, \mu \rangle = \int_E f(x)\mu(dx)$. A topology on $M_h(E)$ can be defined by the convention

$$\mu_n \longrightarrow \mu \text{ in } M_h(E) \qquad \Longleftrightarrow \qquad \langle f, \mu_n \rangle \longrightarrow \langle f, \mu \rangle, \ \ \forall f \in C_h(E). \tag{2.1}$$

Denote by $\mathcal{B}(M_h(E))$ the corresponding Borel-σ-algebra. Let $\mathbb{1}$ stand for the constant function equal to 1. Then $B_{\mathbb{1}}(E)$ is the space of bounded Borel functions while $M_{\mathbb{1}}(E)$ denotes the space of finite Borel measures over E. Using the homeomorphisms $M_h(E) \ni \mu \longmapsto h(x)\mu(dx) \in M_{\mathbb{1}}(E)$ and $B_{\mathbb{1}}(E) \ni f \longmapsto hf \in B_h(E)$ combined with [33, Corollary 1.12] one can show that

$$\mathcal{B}(M_h(E)) = \sigma(\{\mu \longmapsto \langle f, \mu \rangle \mid f \in B_h(E)^+\}). \tag{2.2}$$

Let $\mathcal{P}(M_h(E))$ be the space of all Borel probability measures over $M_h(E)$. For $\rho \in \mathcal{P}(M_h(E))$ the Laplace transform of ρ is defined by

$$\mathcal{L}_\rho(f) = \int_{M_h(E)} e^{-\langle f, \nu \rangle} \rho(d\nu), \qquad f \in B_h(E)^+.$$

We say that $(f_n)_{n\geq 1} \subset B_h(E)$ converges bounded pointwise to some $f \in B_h(E)$, if $\sup_{n\geq 1} \|f_n\|_h < \infty$ and $f_n \longrightarrow f$ pointwise as $n \to \infty$. In this work we will frequently use the following classical properties of the Laplace transform.

Proposition 2.1 *The following assertions hold:*

(a) Let $\rho, \widetilde{\rho} \in \mathcal{P}(M_h(E))$ be such that $\mathcal{L}_\rho(f) = \mathcal{L}_{\widetilde{\rho}}(f)$ holds for all $f \in C_h(E)^+$. Then $\rho = \widetilde{\rho}$.

(b) Let $(\rho_n)_{n\geq 1} \subset \mathcal{P}(M_h(E))$ and $\rho \in \mathcal{P}(M_h(E))$. Then $\rho_n \longrightarrow \rho$ weakly in $\mathcal{P}(M_h(E))$ if and only if $\lim_{n\to\infty} \mathcal{L}_{\rho_n}(f) = \mathcal{L}_\rho(f)$ *for all $f \in C_h(E)^+$.*

(c) Let $\mathcal{L}$ be a functional on $B_h(E)^+$ continuous at $f = 0$ with respect to bounded pointwise convergence. Suppose that there exists $(\rho_n)_{n\geq 1} \subset \mathcal{P}(M_h(E))$ with $\lim_{n\to\infty} \mathcal{L}_{\rho_n}(f) = \mathcal{L}(f)$ *for all $f \in B_h(E)^+$. Then there exists $\rho \in \mathcal{P}(M_h(E))$ such that $\mathcal{L}_\rho = \mathcal{L}$.*

Proof Using the homeomorphisms $M_h(E) \ni \mu(dx) \longmapsto h(x)\mu(dx) \in M_{\mathbb{1}}(E)$ and $B_{\mathbb{1}}(E) \ni f \longmapsto hf \in B_h(E)$ the assertions are particular cases of the properties of Laplace transforms on $M_{\mathbb{1}}(E)$, see e.g. [33, Chapter 1]. □

Below we describe the class of measure-valued branching processes with immigration studied in this work. The spatial motion is described by a process satisfying the following condition:

(A1) Let $p_t^\xi(x, dy)$ be the transition kernel of a conservative Borel right process ξ on E. Suppose that there exists $\alpha \geq 0$ such that

$$\lim_{t\to 0} e^{-\alpha t} \int_E h(y) p_t^\xi(x, dy) = h(x) \tag{2.3}$$

increasingly for each $x \in E$.

Note that each Feller process is a Borel right process. Property Eq. 2.3 states that h is an α-excessive function for the transition semigroup. We refer to [33, Appendix A.3] for additional details on Borel right processes. Below we state a simple sufficient condition for Eq. 2.3. A bounded and Borel measurable function $f : E \longrightarrow [0, \infty)$ is called *finely continuous w.r.t.* ξ, if $t \longmapsto f(\xi_t)$ is a.s. right-continuous on $[0, \infty)$. Let $C^\xi(E)$ be the set of all bounded and Borel measurable functions f which are also finely continuous w.r.t. ξ. Then

$$t \longmapsto p_t^\xi f(x) := \int_E f(y) p_t^\xi(x, dy)$$

is right-continuous for each $x \in E$ and $f \in C^\xi(E)$. For $\beta > 0$ the β-resolvent of ξ is defined by

$$U^\beta f(x) = \int_E e^{-\beta t} P_t f(x) dt, \qquad \forall x \in E, \quad f \in C^\xi(E).$$

Define $D(A) = U^\beta C^\xi(E)$ and for $f = U^\beta g$ with $g \in C^\xi(E)$ let $Af = \beta f - g$. Then $D(A)$ and Af are both independent of β. The operator $(A, D(A))$ defined in this way is called *weak generator* of ξ. It follows from [33, Theorem A.46] that for each $f \in D(A)$ one has

$$p_t^\xi f(x) = f(x) + \int_0^t p_s^\xi Af(x) ds, \qquad t \geq 0, \quad x \in E.$$

The following is a useful sufficient condition for Eq. 2.3.

Remark 2.2 Let $(A, D(A))$ be the weak generator of ξ. If $h \in D(A)$ and there exists $\alpha \geq 0$ such that $Ah \leq \alpha h$, then Eq. 2.3 is satisfied.

Proof Using integration by parts, gives

$$e^{-\alpha t} p_t^{\xi} h(x) = h(x) + \int_0^t e^{-\alpha s} \left(p_s^{\xi} Ah - \alpha h \right)(x) ds.$$

Observing that the integrand in the right-hand side is nonpositive, proves the assertion. □

If E is a locally compact space and $(p_t^{\xi})_{t \geq 0}$ is a Feller semigroup with respect to $C_0(E)$ (space of continuous functions vanishing at infinity), then ξ can be chosen to be càdlàg. Hence $C_0(E) \subset C^{\xi}(E)$ and the weak generator is an extension of the standard generator for strongly continuous semigroups.

Write $M_h(E)^{\circ} = M_h(E) \backslash \{0\}$ where 0 denotes the zero measure. We endow $M_h(E)^{\circ}$ with the restriction of the topology from $M_h(E)$ and the corresponding trace σ-algebra. For $x \in E$ and $f \in B_h(E)^+$ introduce the branching mechanism

$$\begin{aligned} \phi(x, f) \quad &= c(x) f(x)^2 + b(x) f(x) - \textstyle\int_E f(y) \eta(x, dy) \\ &+ \textstyle\int_{M_h(E)^{\circ}} \left(e^{-\langle f, \nu \rangle} - 1 + f(x) \nu(\{x\}) \right) H_1(x, d\nu) \end{aligned}$$

and immigration mechanism

$$\psi(f) = \langle f, \beta \rangle + \int_{M_h(E)^{\circ}} \left(1 - e^{-\langle f, \nu \rangle} \right) H_2(d\nu). \tag{2.4}$$

The corresponding parameters are supposed to satisfy the following conditions:

(A2) $b \in B_{\mathbb{1}}(E)$, $ch \in B_{\mathbb{1}}(E)^+$, η is a σ-finite kernel on E and $H_1(x, d\nu)$ is a σ-finite kernel from E to $M_h(E)^{\circ}$ satisfying, for each $x \in E$,

$$\int_E h(y) \eta(x, dy) + \int_{M_h(E)^{\circ}} \left(\langle h, \nu \rangle \wedge \langle h, \nu \rangle^2 + \langle h, \nu_x \rangle \right) H_1(x, d\nu) \leq C h(x),$$

for some constant $C > 0$, where ν_x denotes the restriction of $\nu(dy)$ to $E \backslash \{x\}$.

(A3) $\beta \in M_h(E)$ and H_2 is a Borel measure on $M_h(E)^{\circ}$ satisfying

$$\int_{M_h(E)^{\circ}} 1 \wedge \langle h, \nu \rangle H_2(d\nu) < \infty. \tag{2.5}$$

The next result provides the construction of the measure-valued branching processes with immigration.

Theorem 2.3 *Suppose that conditions (A1) – (A3) are satisfied. Then the following assertions hold:*

(a) For each $f \in B_h(E)^+$ there exists a unique nonnegative solution $\mathbb{R}_+ \times E \ni (t, x) \longmapsto v_t(x, f)$ to

$$v_t(x, f) = \int_E f(y) p_t^{\xi}(x, dy) - \int_0^t \int_E \phi(y, v_s(\cdot, f)) p_{t-s}^{\xi}(x, dy) ds, \tag{2.6}$$

so that $t \longmapsto \| v_t(\cdot, f) \|_h$ is bounded on each bounded interval $[0, T]$.

(b) Letting $V_t f(x) = v_t(x, f)$, we find that $(V_t)_{t \geq 0}$ is a cumulant semigroup in the sense that $V_t V_s = V_{t+s}$ for all $t, s \geq 0$ and

$$V_t f(x) = \int_E f(y) \lambda_t(x, dy) + \int_{M_h(E)^{\circ}} \left(1 - e^{-\langle f, \nu \rangle} \right) L_t(x, d\nu), \tag{2.7}$$

where $\lambda_t(x, dy)$ and $L_t(x, d\nu)$ are transition kernels such that for each $t \geq 0$ there exists a constant $C_t > 0$ satisfying

$$\int_E h(y)\lambda_t(x, dy) + \int_{M_h(E)^\circ} 1 \wedge \langle h, \nu\rangle L_t(x, d\nu) \leq C_t h(x), \qquad x \in E.$$

(c) There exists a unique Markov kernel $P_t(\mu, d\nu)$ whose Laplace transform is, for $t \geq 0$, $\mu \in M_h(E)$ and $f \in B_h(E)^+$, given by

$$\int_{M_h(E)} e^{-\langle f,\nu\rangle} P_t(\mu, d\nu) = \exp\left(-\langle V_t f, \mu\rangle - \int_0^t \psi(V_s f) ds\right). \tag{2.8}$$

This result can be obtained from [33], details are postponed to the Appendix A.

Remark 2.4 Let $(A, D(A))$ be the weak generator of ξ. Then Eq. 2.6 is a mild formulation of Eq. 1.1.

In this work we will always assume that conditions (A1) – (A3) are satisfied and call the corresponding Markov process (ξ, ϕ, ψ)-superprocess. Under some additional conditions one may also identify the generator of this process in terms of a martingale problem. The latter one is for some class of functions $F : M_h(E) \longrightarrow \mathbb{R}$ formally given by

$$\begin{aligned}\mathcal{A}F(\mu) = & \int_E \mu(dx)\left(AF'(\mu; x) - b(x)F'(\mu; x) + \int_E F'(\mu; y)\eta(x, dy) + c(x)F''(\mu; x)\right) \\ & + \int_E \mu(dx) \int_{M_h(E)^\circ} \left(F(\mu+\nu) - F(\mu) - F'(\mu; x)\nu(\{x\})\right) H_1(x, d\nu) \\ & + \int_E F'(\mu; x)\beta(dx) + \int_{M_h(E)^\circ} (F(\mu+\nu) - F(\mu)) H_2(d\nu)\end{aligned}$$

where the variational derivaties are defined by

$$F'(\mu; x) = \lim_{\varepsilon\to 0} \frac{F(\mu+\varepsilon\delta_x) - F(\mu)}{\varepsilon}, \; F''(\mu; x) = \lim_{\varepsilon\to 0} \frac{F'(\mu+\varepsilon\delta_x; x) - F'(\mu; x)}{\varepsilon}.$$

Since we will not make use of the corresponding martingale problem characterization, we refer the reader to [33, Section 9] for additional details.

3 Behavior of First Moment

In order to study the first moment of $P_t(\mu, d\nu)$ it is reasonable to rewrite the branching mechanism to

$$\begin{aligned}\phi(x, f) = & \; c(x)f(x)^2 + b(x)f(x) - \int_E f(y)\gamma(x, dy) \\ & + \int_{M_h(E)^\circ} \left(e^{-\langle f,\nu\rangle} - 1 + \langle f, \nu\rangle\right) H_1(x, d\nu),\end{aligned}$$

where $\gamma(x, dy)$ is defined by

$$\gamma(x, dy) = \eta(x, dy) + \int_{M_h(E)^\circ} \nu_x(dy) H_1(x, d\nu).$$

In this way the contribution of the branching to the first moment is encoded in the properties of the kernel γ and its associated transition operator

$$\Gamma f(x) = \int_E f(y)\gamma(x,dy), \qquad f \in B_h(E).$$

Since the operator B defined by $Bf(x) = \Gamma f(x) - b(x)f(x)$ is bounded on $B_h(E)$, it can be shown that there exists a unique semigroup $(R_t)_{t\geq 0}$ on $B_h(E)$ which satisfies

$$R_t f(x) = p_t^{\xi} f(x) + \int_0^t p_s^{\xi} B R_{t-s} f(x) ds, \qquad x \in E. \tag{3.1}$$

Such semigroup can be obtained by iterating Eq. 3.1 and it can be shown that

$$\sup_{t\in[0,T]} \sup_{\|f\|_h \leq 1} \|R_t f\|_h < \infty, \qquad T > 0. \tag{3.2}$$

Finally, since Γ maps $B_h(E)^+$ to itself, it can be shown that $R_t f \geq 0$ whenever $f \in B_h(E)^+$. Let us stress that $(R_t)_{t\geq 0}$ and $(p_t^{\xi})_{t\geq 0}$ need not to be strongly continuous neither on $B_h(E)$ nor on $C_h(E)$. Moreover, $(R_t)_{t\geq 0}$ does not need to be conservative, i.e. $R_t 1 \neq 1$ is allowed. We will need the following simple properties of the semigroup $(R_t)_{t\geq 0}$.

Lemma 3.1 *The semigroup $(R_t)_{t\geq 0}$ satisfies for each $t \geq 0$, $x \in E$ and $f \in B_h(E)^+$*

$$R_t f(x) = \int_E f(y)\lambda_t(x,dy) + \int_{M_h(E)^\circ} \langle f, \nu\rangle L_t(x,d\nu). \tag{3.3}$$

and

$$\frac{1}{\varepsilon} V_t(\varepsilon f)(x) \nearrow R_t f(x), \qquad \varepsilon \searrow 0.$$

The proof of this lemma is postponed to the Appendix A. Set

$$r_t(x,dy) = \lambda_t(x,dy) + \int_{M_h(E)^\circ} \nu(dy) L_t(x,d\nu).$$

Using Eq. 3.3 one can show that for each $f \in B_h(E)$ and $t \geq 0$

$$R_t f(x) = \int_E f(y) r_t(x,dy), \qquad x \in E. \tag{3.4}$$

The action of the adjoint semigroup to $(R_t)_{t\geq 0}$ acting on $M_h(E)$ is given by

$$R_t^* \nu(dy) = \int_E r_t(x,dy)\nu(dx), \qquad t \geq 0$$

and describes the evolution of the first moment for the (ξ, ϕ, ψ)-superprocess.

Proposition 3.2 *Suppose that H_2 satisfies*

$$\int_{\{\nu \in M_h(E)^\circ \,:\, \langle h,\nu\rangle > 1\}} \langle h, \nu\rangle H_2(d\nu) < \infty. \tag{3.5}$$

Then for each $\mu \in M_h(E)$ one has

$$\int_{M_h(E)} \nu(dx) P_t(\mu, d\nu) = R_t^* \mu(dx) + \int_0^t R_s^* a(dx) ds,$$

where both integrals are understood in a weak sense in $M_h(E)$, and

$$a(dx) = \beta(dx) + \int_{M_h(E)^\circ} \nu(dx) H_2(d\nu).$$

If $h \equiv 1$, then the assertion follows from [33, Proposition 9.11]. A similar result can be also deduced in the case $h \neq 1$. Since we could not find a precise reference, for convenience of the reader a proof is given in the Appendix A. This result suggests to relate the long-time behavior of (ξ, ϕ, ψ)-superprocesses with the stability of the semigroup $(R_t)_{t \geq 0}$.

Definition 3.3 A (ξ, ψ, ϕ)-superprocess is called subcritical, if the semigroup $(R_t)_{t \geq 0}$ satisfies $\lim_{t \to \infty} \langle R_t h, \mu \rangle = 0$ for each $\mu \in M_h(E)$.

Note that the (ξ, ψ, ϕ)-superprocess is subcritical, if there exists a constant $C > 0$ such that the associated semigroup $(R_t)_{t \geq 0}$ satisfies

$$R_t h(x) \leq C h(x), \quad \text{and} \quad \lim_{t \to \infty} R_t h(x) = 0, \qquad x \in E, \quad t \geq 0.$$

For a subcritical superprocess the mass located inside the system gets extinct in the limit $t \to \infty$. However, as additional mass also immigrates into the system, it is reasonable to expect that the mean mass of the process is positive in the limit $t \to \infty$. The latter one being formally described below by the first moment of the superprocess.

Corollary 3.4 *Let $P_t(\mu, d\nu)$ be the Markov kernel of a subcritical (ξ, ϕ, ψ)-superprocess satisfying (3.5). Then for each $\mu \in M_h(E)$ and each $f \in B_h(E)^+$ one has*

$$\lim_{t \to \infty} \int_{M_h(E)} \langle f, \nu \rangle P_t(\mu, d\nu) = \int_0^\infty \langle R_s f, a \rangle ds \in [0, \infty].$$

If $\int_0^\infty \langle R_s h, a \rangle ds < \infty$, then this equality also holds for $f \in B_h(E)$.

Example 3.5 If $\Gamma = 0$ and $A = 0$, i.e. $p_t^\xi(x, dy) = \delta_x(dy)$, then $R_t f(x) = e^{-tb(x)} f(x)$. Suppose that $b(x) > 0$ for each $x \in E$, then $\lim_{t \to \infty} \langle R_t h, \mu \rangle = 0$ and

$$\int_0^\infty \langle R_s h, \mu \rangle ds = \int_E \int_0^\infty h(x) e^{-sb(x)} ds \mu(dx) = \int_E \frac{h(x)}{b(x)} \mu(dx)$$

holds for each $\mu \in M_h(E)$.

Above proposition motivates the following definition.

Definition 3.6 A (ξ, ψ, ϕ)-superprocess is called exponentially subcritical, if the semigroup $(R_t)_{t \geq 0}$ is uniformly stable, i.e., $\lim_{t \to \infty} \| R_t h \|_h = 0$.

Lemma 3.7 *The (ξ, ψ, ϕ)-superprocess is exponentially subcritical if and only if there exist constants $C, \delta > 0$ such that, for each $f \in B_h(E)^+$,*

$$R_t f(x) \leq \| f \|_h C h(x) e^{-\delta t}, \qquad t \geq 0, \quad x \in E. \tag{3.6}$$

Proof If $(R_t)_{t \geq 0}$ is strongly continuous, then this assertion is a direct consequence of classical semigroup theory, see [9, Chapter V, Proposition 1.2]. Since in our case $(R_t)_{t \geq 0}$ is not necessarily strongly continuous, we provide a short proof. Using Eq. 3.6 for $f = h$ shows that the (ξ, ψ, ϕ)-superprocess is exponentially subcritical. Conversely, suppose that the (ξ, ψ, ϕ)-superprocess is exponentially subcritical. Take $t_0 > 0$ such that $\| R_{t_0} h \|_h < 1$. Then using the positivity of $(R_t)_{t \geq 0}$ we find for each $f \in B_h(E)^+$ and $x \in E$ that

$$R_{t_0} f(x) \leq \| f \|_h R_{t_0} h(x) \leq \| f \|_h h(x) \| R_{t_0} h \|_h,$$

and hence $\|R_{t_0}\|_{op} \leq \|R_{t_0}h\|_h$ where $\|R_{t_0}\|_{op}$ denotes the operator norm on $B_h(E)$. For $t \geq 0$, write $t = nt_0 + r$ where $n \in \mathbb{N}_0$ and $r \in [0, t_0)$. Then using the semigroup property gives for each $f \in B_h(E)^+$

$$\|R_t f\|_h \leq \|R_{t_0}\|_{op}^n \|R_r f\|_h \leq \left(\sup_{s\in[0,t_0]} \|R_s\|_{op} \right) \|f\|_h \omega^n$$

where $\omega = \|R_{t_0}h\|_h$ and $\sup_{s\in[0,t_0]} \|R_s\|_{op}$ is finite due to Eq. 3.2. If $\omega = 0$, then nothing has to be shown. If $\omega \in (0, 1)$, then the assertion follows from

$$\omega^n = \omega^{\frac{t-r}{t_0}} = \left(\omega^{1/t_0}\right)^t \left(\frac{1}{\omega}\right)^{r/t_0} \leq e^{t\frac{\ln(\omega)}{t_0}} \frac{1}{\omega} = \frac{e^{-t\frac{\ln(1/\omega)}{t_0}}}{\omega}.$$

□

Remark 3.8 If $(R_t)_{t\geq 0}$ satisfies Eq. 3.6, then $\langle R_t f, \mu\rangle \leq C\|f\|_h e^{-\delta t}\langle h, \mu\rangle$ holds for each $\mu \in M_h(E)$ and

$$\langle R_s f, a\rangle \leq C\|f\|_h \langle h, \beta\rangle e^{-\delta s} + C\|f\|_h e^{-\delta s} \int_{M_h(E)^\circ} \langle h, \nu\rangle H_2(d\nu)$$

holds for each $f \in B_h(E)^+$.

The next proposition provides a Lyapunov-type condition such that the process is exponentially subcritical.

Proposition 3.9 *Suppose that h belongs to the domain of the weak generator of* $(A, D(A))$. *Then the following assertions hold:*

(a) If there exists $\delta > 0$ *such that*

$$\Gamma h(x) \leq b(x)h(x) - \delta h(x), \qquad x \in E,$$

then $R_t h(x) \leq h(x)e^{-(\delta-\alpha)t}$ *holds for* $x \in E$ *and* $t \geq 0$ *with* α *given as in condition (A1).*

(b) If there exists $\delta > 0$ *such that*

$$Ah(x) + \Gamma h(x) \leq b(x)h(x) - \delta h(x), \qquad x \in E, \tag{3.7}$$

then $R_t h(x) \leq h(x)e^{-\delta t}$ *holds for* $x \in E$ *and* $t \geq 0$.

Proof Using the product rule and that h belongs to the domain of the weak generator, we find that

$$e^{\delta t} R_t h(x) = h(x) + \int_0^t e^{\delta s} R_s \left(Ah - bh + \Gamma h + \delta h\right)(x) ds.$$

Under condition (a), we may use the positivity of R_s, to find that

$$e^{\delta t} R_t h(x) \leq h(x) + \int_0^t e^{\delta s} R_s Ah(x) ds.$$

Hence $e^{\delta t} R_t h(x) \leq p_t^{\xi} h(x) \leq e^{\alpha t} h(x)$, which proves the first assertion. Under condition (b), we find directly $e^{\delta t} R_t h(x) \leq h(x)$ which proves the assertion. □

4 Invariant Probability Measures

In this section we provide a sufficient condition for the existence and uniqueness of an invariant probability measure for a subcritical (ξ, ϕ, ψ)-superprocess with Markov kernel $P_t(\mu, d\nu)$. Recall that $\pi \in \mathcal{P}(M_h(E))$ is an invariant probability measure, if

$$\int_E P_t(\mu, d\nu)\pi(d\mu) = \pi(d\nu), \qquad t \geq 0. \tag{4.1}$$

The following simple observation is our main tool for the study of limiting distributions.

Proposition 4.1 *Let $P_t(\mu, d\nu)$ be the transition kernel of a (ξ, ϕ, ψ)-superprocess. The following are equivalent:*

(i) There exists $\pi \in \mathcal{P}(M_h(E))$ such that $P_t(\mu, d\nu) \longrightarrow \pi$ weakly as $t \to \infty$ for each $\mu \in M_h(E)$.

(ii) For each $\mu \in M_h(E)$ it holds that

$$\lim_{t\to\infty} \langle V_t h, \mu\rangle = 0, \qquad \int_0^\infty \psi(V_s h)ds < \infty,$$

and $f \longmapsto \int_0^\infty \psi(V_s f)ds$ is continuous at $f = 0$ with respect to bounded pointwise convergence.

In this case π is the unique invariant probability measure and its Laplace transform is given by

$$\mathcal{L}_\pi(f) = \exp\left(-\int_0^\infty \psi(V_s f)ds\right), \qquad f \in B_h(E)^+. \tag{4.2}$$

Proof $(i) \Rightarrow (ii)$ The characterization of weak convergence in terms of Laplace transforms shows that (i) is equivalent to $\lim_{t\to\infty} \mathcal{L}_{P_t(\mu,\cdot)}(f) = \mathcal{L}_\pi(f)$ for each $f \in C_h(E)^+$. Using this fact for $\mu = 0$ combined with $\psi \geq 0$ gives

$$\begin{aligned}\mathcal{L}_\pi(f) &= \lim_{t\to\infty} \mathcal{L}_{P_t(0,\cdot)}(f) \\ &= \lim_{t\to\infty} \exp\left(-\int_0^t \psi(V_s f)ds\right) = \exp\left(-\int_0^\infty \psi(V_s f)ds\right).\end{aligned}$$

This shows that $\int_0^\infty \psi(V_s h)ds < \infty$, $f \longmapsto \int_0^\infty \psi(V_s f)ds$ is continuous at $f = 0$ and Eq. 4.2 holds. Hence for each $\mu \in M_h(E)$

$$\begin{aligned}\exp\left(-\int_0^\infty \psi(V_s f)ds\right) &= \mathcal{L}_\pi(f) \\ &= \lim_{t\to\infty} \mathcal{L}_{P_t(\mu,\cdot)}(f) \\ &= \exp\left(-\lim_{t\to\infty}\langle V_t f, \mu\rangle - \lim_{t\to\infty}\int_0^t \psi(V_s f)ds\right)\end{aligned}$$

which readily yields $\lim_{t\to\infty}\langle V_t f, \mu\rangle = 0$. This proves (ii).

$(ii) \Rightarrow (i)$ In this case it is clear that $\lim_{t\to\infty} \mathcal{L}_{P_t(\mu,\cdot)}(f) = \exp\left(-\int_0^\infty \psi(V_s f)ds\right)$ and that the right-hand side is continuous at $f = 0$ with respect to bounded pointwise convergence. This readily yields (i) as well as Eq. 4.2.

It remains to show that π is the unique invariant probability measure. In order to show that π is invariant, we use a similar argument to [23]. Fix $f \in B_h(E)^+$ and $t \geq 0$. Then using $V_s V_t = V_{s+t}$ for $s, t \geq 0$ yields

$$\begin{aligned}
&\int_{M_h(E)} e^{-\langle f,\nu\rangle} \left(\int_{M_h(E)} P_t(\mu, d\nu)\pi(d\mu) \right) \\
&= \int_{M_h(E)} \exp\left(-\langle V_t f, \mu\rangle - \int_0^t \psi(V_s f)ds \right) \pi(d\mu) \\
&= \exp\left(-\int_0^t \psi(V_s f)ds \right) \exp\left(-\int_0^\infty \psi(V_s V_t f)ds \right) \\
&= \exp\left(-\int_0^t \psi(V_s f)ds \right) \exp\left(-\int_t^\infty \psi(V_s f)ds \right) \\
&= \exp\left(-\int_0^\infty \psi(V_s f)ds \right) = \mathcal{L}_\pi(f).
\end{aligned}$$

Hence by uniqueness of Laplace transforms (see Proposition 2.1) we conclude that π is invariant. Next we prove that π is the unique invariant probability measure. Let π' be another invariant probability measure. For each $t \geq 0$ we obtain from Eq. 4.1

$$\mathcal{L}_{\pi'}(f) = \int_{M_h(E)} \left(\int_{M_h(E)} e^{-\langle f,\nu\rangle} P_t(\mu, d\nu) \right) \pi'(d\mu).$$

Taking the limit $t \to \infty$ gives

$$\mathcal{L}_{\pi'}(f) = \exp\left(-\int_0^\infty \psi(V_s f)ds \right) = \mathcal{L}_\pi(f).$$

Since f was arbitrary, we conclude that $\pi = \pi'$. □

Without immigration (i.e. $\psi \equiv 0$) one has $\mathcal{L}_\pi(f) \equiv 1$ and hence $\pi = \delta_0$, where 0 denotes the zero measure. This is not surprising as we work with subcritical superprocesses. From this point of view, when working with subcritical superprocesses immigration is necessary for the existence of a nontrivial invariant measure. A similar effect was recently observed in [17] for birth-and-death processes in the continuum. Below we discuss a set of conditions for which Proposition 4.1 can be applied.

Theorem 4.2 *Consider a (ξ, ϕ, ψ)-superprocess with associated semigroup $(R_t)_{t\geq 0}$. Then the following assertions hold:*

(a) Suppose that the (ξ, ϕ, ψ)-superprocess is subcritical, satisfies (3.5) and that

$$\int_0^\infty \langle R_s h, a\rangle ds < \infty \tag{4.3}$$

holds for $a(dx) = \beta(dx) + \int_{M_h(E)^\circ} \nu(dx)H_2(d\nu)$. Then condition (ii) from Proposition 4.1 is satisfied.

(b) Suppose that the (ξ, ϕ, ψ)-superprocess is exponentially subcritical and

$$\int_{\{\nu\in M_h(E)^\circ \,:\, \langle h,\nu\rangle>1\}} \log(\langle h, \nu\rangle) H_2(d\nu) < \infty. \tag{4.4}$$

Then condition (ii) from Proposition 4.1 is satisfied.

Proof Let $Q_t(\mu, d\nu)$ be the Markov kernel of a $(\xi, \phi, \psi = 0)$-superprocess given by Eq. 2.8. By Jensen's inequality applied to the convex function $t \longmapsto e^{-t}$ we obtain

$$\begin{aligned}\exp\left(-\langle V_t f, \mu\rangle\right) &= \int_{M_h(E)} e^{-\langle f,\nu\rangle} Q_t(\mu, d\nu)\\ &\geq \exp\left(-\int_{M_h(E)} \langle f, \nu\rangle Q_t(\mu, d\nu)\right) = \exp\left(-\langle R_t f, \mu\rangle\right),\end{aligned}$$

where the last equality follows from Proposition 3.2. From this we conclude that $V_t f(x) \leq R_t f(x)$ holds for all $x \in E$.

(a) Let $f \in B_h(E)^+$, then

$$\psi(V_s f) \leq \left(\langle R_s f, \beta\rangle + \int_{M_h(E)^\circ} \langle R_s f, \nu\rangle H_2(d\nu)\right) = \langle R_s f, a\rangle.$$

This shows that $\int_0^\infty \psi(V_s f)ds < \infty$. To prove the desired continuity at $f = 0$, we let $(f_n)_{n\in\mathbb{N}} \subset B_h(E)^+$ be such that $f_n \longrightarrow f$ bounded pointwise. Then

$$\begin{aligned}\int_0^\infty \psi(V_s f_n)ds &\leq \int_0^\infty \langle R_s f_n, a\rangle ds\\ &= \int_0^\infty \int_E \int_E f_n(y) r_s(x, dy) a(dx) ds.\end{aligned}$$

Observe that the integrand converges to zero as $n \to \infty$. Since $f_n(y) \leq \sup_{n\in\mathbb{N}} \|f_n\|_h h(y)$ and

$$\int_0^\infty \int_E \int_E f_n(y) r_s(x, dy) a(dx) ds \leq \sup_{n\in\mathbb{N}} \|f_n\|_h \int_0^\infty \langle R_s h, a\rangle ds < \infty$$

the dominated convergence theorem implies that $\lim_{n\to\infty} \int_0^\infty \psi(V_s f_n)ds = 0$, which proves the assertion.

(b) Let $f \in B_h(E)^+$, $s \geq 0$ and $\nu \in M_h(E)^\circ$. Denote by $C > 0$ a generic constant which may vary from line to line and is independent of s, ν, f. Then we obtain

$$\begin{aligned}\left(1 - e^{-\langle V_s f,\nu\rangle}\right) &\leq \min\{1, \langle V_s f, \nu\rangle\}\\ &\leq \mathbb{1}_{\{\langle h,\nu\rangle\leq 1\}} \langle V_s f, \nu\rangle + C \mathbb{1}_{\{\langle h,\nu\rangle>1\}} \log(1 + \langle V_s f, \nu\rangle)\\ &=: I_1 + I_2.\end{aligned}$$

For the first term we obtain

$$I_1 \leq \mathbb{1}_{\{\langle h,\nu\rangle\leq 1\}} C \|f\|_h e^{-\delta s} \langle h, \nu\rangle.$$

The second term is estimated by

$$\begin{aligned}I_2 &\leq \mathbb{1}_{\{\langle h,\nu\rangle>1\}} \log(1 + C\|f\|_h e^{-\delta s} \langle h, \nu\rangle)\\ &\leq \mathbb{1}_{\{\langle h,\nu\rangle>1\}} C\|f\|_h e^{-\delta s} \left(1 + \log(1 + \langle h, \nu\rangle)\right),\end{aligned}$$

where we have used, for $x = C\|f\|_h e^{-\delta s}$ and $y = \langle h, \nu\rangle$, the elementary estimate

$$\begin{aligned}\log(1 + xy) &\leq C \log(1 + x) \log(1 + y) + C \min\{\log(1 + x), \log(1 + y)\}\\ &\leq Cx \left(1 + \log(1 + y)\right), \qquad (4.5)\end{aligned}$$

see the Appendix A in [16] for a proof. Inserting this into the definition of ψ gives

$$\begin{aligned}\psi(V_s f) &= \langle V_s f, \beta\rangle + \int_{M_h(E)^\circ} \left(1 - e^{-\langle V_s f, \nu\rangle}\right) H_2(d\nu) \\ &\le C\|f\|_h e^{-\delta s}\langle h, \beta\rangle + C\|f\|_h e^{-\delta s} \int_{\{\langle h,\nu\rangle \le 1\}} \langle h, \nu\rangle H_2(d\nu) \\ &\quad + C\|f\|_h e^{-\delta s} \int_{\{\langle h,\nu\rangle > 1\}} (1 + \log(\langle h, \nu\rangle))\, H_2(d\nu).\end{aligned}$$

Using $\log(1+y) \le \log(2y) = \log(2) + \log(y)$ for $y > 1$, Eqs. 4.4, and 2.5 we find that the right-hand side is finite. This proves $\int_0^\infty \psi(V_s f)ds \le C\delta^{-1}\|f\|_h < \infty$. The continuity of $f \longmapsto \int_0^\infty \psi(V_s f)ds$ at $f = 0$ with respect to bounded pointwise convergence can be shown similarly to the proof given in part (a). □

A similar formula to Eq. 4.2 was obtained by Stannat [41, 42] for the case of a compact location space E with $H_2 = 0$, ϕ given by Eq. 1.4 and $h \equiv 1$. Although the proof of Theorem 4.2 does not look very difficult, the particular case $E = \{1, \dots, d\}$ with general (nonlocal) branching and immigration mechanism was only recently established in [23] where affine processes on the canonical state space have been studied.

5 Ergodicity and Convergence Rate

In this section we study the convergence rate for $P_t(\mu, \cdot) \longrightarrow \pi$ in two different distances on $\mathcal{P}(M_h(E))$.

5.1 Distances on $\mathcal{P}(M_h(E))$

Let $\|\cdot\|_{TV}$ be the total variation distance on the space of finite measures $M_1(E)$. Using the homeomorphism $M_h(E) \ni \mu \longmapsto h(x)\mu(dx) \in M_{\mathbb{1}}(E)$, we transport this distance to $M_h(E)$ by setting

$$\|\mu - \widetilde{\mu}\|_{TV,h} = \|h\mu - h\widetilde{\mu}\|_{TV}.$$

Note that $\|\mu\|_{TV,h} = \langle h, \mu\rangle$ for $\mu \in M_h(E)$. Using again the homeomorphism $M_h(E) \ni \mu \longmapsto h(x)\mu(dx) \in M_{\mathbb{1}}(E)$ combined with [33, Corollary 1.12] one can show that the Borel-σ-algebra generated by this weighted total variation distance coincides with Eq. 2.2. Hence $\mathcal{P}(M_h(E))$ remains the same although the topologies generated by Eq. 2.1 and the weighted total variation distance do not coincide.

Given $\rho, \widetilde{\rho} \in \mathcal{P}(M_h(E))$, we call a Borel probability measure H on the product space $M_h(E) \times M_h(E)$ a coupling of $(\rho, \widetilde{\rho})$ if its marginals are given by ρ and $\widetilde{\rho}$, respectively. We denote the set of all such couplings by $\mathcal{H}(\rho, \widetilde{\rho})$. Below we consider on $\mathcal{P}(M_h(E))$ the following distances:

- The Laplace distance defined by

$$d_{\mathcal{L}}(\rho, \widetilde{\rho}) = \sup_{f \in B_h(E)^+ \setminus \{0\}} \frac{|\mathcal{L}_\rho(f) - \mathcal{L}_{\widetilde{\rho}}(f)|}{\|f\|_h} \in [0, \infty].$$

- The Wasserstein-1-distance defined by

$$W_1(\rho, \widetilde{\rho}) = \inf\left\{ \int_{M_h(E)\times M_h(E)} \|\mu - \widetilde{\mu}\|_{TV,h} H(d\mu, d\widetilde{\mu}) :\ H \in \mathcal{H}(\rho, \widetilde{\rho}) \right\} \in [0, \infty].$$

Let $\mathcal{P}_1(M_h(E))$ be the subspace of $\mathcal{P}(M_h(E))$ consisting of all probability measures with finite h-moment, i.e.

$$\mathcal{P}_1(M_h(E)) = \left\{ \rho \in \mathcal{P}(M_h(E)) \ : \ \int_{M_h(E)} \langle h, \mu \rangle \rho(d\mu) < \infty \right\}.$$

Using $\|\mu - \widetilde{\mu}\|_{TV,h} \leq \|\mu\|_{TV,h} + \|\widetilde{\mu}\|_{TV,h} = \langle h, \mu \rangle + \langle h, \widetilde{\mu} \rangle$ we find that

$$W_1(\rho, \widetilde{\rho}) \leq \int_{M_h(E)} \langle h, \mu \rangle \rho(d\mu) + \int_{M_h(E)} \langle h, \mu \rangle \widetilde{\rho}(d\mu)$$

and hence W_1 is finite on $\mathcal{P}_1(M_h(E))$. Finally, observe that for all $\rho, \widetilde{\rho} \in \mathcal{P}_1(M_h(E))$, $f \in B_h(E)^+$ and $H \in \mathcal{H}(\rho, \widetilde{\rho})$ one has

$$\begin{aligned} |\mathcal{L}_\rho(f) - \mathcal{L}_{\widetilde{\rho}}(f)| &\leq \int_{M_h(E)\times M_h(E)} |\langle f, \mu - \widetilde{\mu} \rangle| H(d\mu, d\widetilde{\mu}) \\ &\leq \|f\|_h \int_{M_h(E)\times M_h(E)} \|\mu - \widetilde{\mu}\|_{TV,h} H(d\mu, d\widetilde{\mu}). \end{aligned}$$

This gives $d_{\mathcal{L}}(\rho, \widetilde{\rho}) \leq W_1(\rho, \widetilde{\rho})$ and hence $d_{\mathcal{L}}$ is finite on $\mathcal{P}_1(M_h(E))$. The relationship for convergence in these two distances is stated below.

Proposition 5.1 *Let $(\rho_n)_{n\in\mathbb{N}} \subset \mathcal{P}_1(M_h(E))$ and $\rho \in \mathcal{P}_1(M_h(E))$. The following are equivalent:*

(a) $\rho_n \longrightarrow \rho$ weakly and

$$\lim_{n\to\infty} \int_{M_h(E)} \langle h, \mu \rangle \rho_n(d\mu) = \int_{M_h(E)} \langle h, \mu \rangle \rho(d\mu).$$

(b) $\rho_n \longrightarrow \rho$ in W_1.

(c) $\rho_n \longrightarrow \rho$ in $d_{\mathcal{L}}$.

Proof The equivalence between (a) and (b) is a consequence of [43, Corollary 6.9]. The implication (b) implies (c) follows from the inequality $d_{\mathcal{L}}(\rho, \widetilde{\rho}) \leq W_1(\rho, \widetilde{\rho})$. Let us prove the implication (c) implies (a). Using the particular choice $f = \varepsilon h$ with $\varepsilon > 0$ in the supremum in the definition in $d_{\mathcal{L}}$ gives

$$d_{\mathcal{L}}(\rho, \widetilde{\rho}) \geq \frac{|\mathcal{L}_\rho(\varepsilon h) - \mathcal{L}_{\widetilde{\rho}}(varepsilonh)|}{\varepsilon} = \left| \int_{M_h(E)} \frac{e^{-\varepsilon\langle h,\mu\rangle} - e^{-\varepsilon\langle h,\widetilde{\mu}\rangle}}{\varepsilon} H(d\mu, d\widetilde{\mu}) \right|$$

for any $H \in \mathcal{H}(\rho, \widetilde{\rho})$. Taking the limit $\varepsilon \to 0$ gives

$$\left| \int_{M_h(E)} \langle h, \mu \rangle \rho(d\mu) - \int_{M_h(E)} \langle h, \mu \rangle \widetilde{\rho}(d\mu) \right| \leq d_{\mathcal{L}}(\rho, \widetilde{\rho}).$$

Hence the implication (c) implies (a) follows from the characterization of weak convergence by Laplace transforms (see Proposition 2.1) and the above inequality in order to also control the first h-moment. □

While the notions of convergence with respect to $d_{\mathcal{L}}$ and W_1 coincide, based on the construction of (ξ, ϕ, ψ)-superprocesses it is natural to measure the rate of convergence in the distance $d_{\mathcal{L}}$. Surprisingly enough, the distance $d_{\mathcal{L}}$ was to our knowledge not used in the literature so far.

5.2 Convergence Rates

For $\rho \in \mathcal{P}(M_h(E))$ we let

$$P_t^*\rho(d\nu) = \int_{M_h(E)} P_t(\mu, d\nu)\rho(d\mu), \qquad t \geq 0. \tag{5.1}$$

It describes the distribution of the (ξ, ϕ, ψ)-superprocess at time $t \geq 0$ when its law at initial time $t = 0$ is given by ρ. Clearly $\pi \in \mathcal{P}(M_h(E))$ is an invariant probability measure if and only if $P_t^*\pi = \pi$ holds for all $t \geq 0$. The following is our first main result of this section.

Theorem 5.2 *Suppose that the (ξ, ϕ, ψ)-superprocess is subcritical, H_2 satisfies (3.5), and* Eq. 4.3. *Then the unique invariant probability measure satisfies $\pi \in \mathcal{P}_1(M_h(E))$ and*

$$W_1(P_t^*\rho, \pi) \leq \int_{M_h(E)} \langle R_t h, \mu\rangle \rho(d\mu) + \int_{M_h(E)} \langle R_t h, \mu\rangle \pi(d\mu), \qquad t \geq 0 \tag{5.2}$$

holds for all $t \geq 0$ and $\rho \in \mathcal{P}_1(M_h(E))$. Moreover, one has

$$\int_{M_h(E)} \langle f, \mu\rangle \pi(d\mu) = \int_0^\infty \langle R_s f, a\rangle ds, \qquad f \in B_h(E). \tag{5.3}$$

Proof Using first $P_t(\mu, \cdot) \longrightarrow \pi$ weakly as $t \to \infty$ combined with the Lemma of Fatou (see [13, Appendix, Proposition 1.1]) gives

$$\begin{aligned} \int_{M_h(E)} \langle h, \nu\rangle \pi(d\nu) &\leq \liminf_{t\to\infty} \int_{M_h(E)} \langle h, \nu\rangle P_t(\mu, d\nu) \\ &= \liminf_{t\to\infty} \left(\langle R_t h, \mu\rangle + \int_0^t \langle R_s h, a\rangle \right) = \int_0^\infty \langle R_s h, a\rangle ds < \infty, \end{aligned}$$

where we have used Proposition 3.2. This proves $\pi \in \mathcal{P}_1(M_h(E))$. Denote by $Q_t(\mu, d\nu)$ the transition probabilities of the $(\xi, \phi, \psi = 0)$-superprocess. From Eq. 2.8 we obtain $P_t(\mu, \cdot) = Q_t(\mu, \cdot) * P_t(0, \cdot)$. Using first the convexity of the Wasserstein distance (see [43, Theorem 4.8]) and then Lemma A.5 yields for any coupling $H \in \mathcal{H}(\rho, \pi)$

$$\begin{aligned} W_1(P_t^*\rho, \pi) &= W_1(P_t^*\rho, P_t^*\pi) \\ &\leq \int_{M_h(E)\times M_h(E)} W_1(P_t(\mu, \cdot), P_t(\widetilde{\mu}, \cdot)) H(d\mu, d\widetilde{\mu}) \\ &\leq \int_{M_h(E)\times M_h(E)} W_1(Q_t(\mu, \cdot), Q_t(\widetilde{\mu}, \cdot)) H(d\mu, d\widetilde{\mu}). \end{aligned}$$

Estimating the Wasserstein distance from above by the particular choice of coupling $Q_t(\mu, d\nu)Q_t(\widetilde{\mu}, d\widetilde{\nu})$ and then using $\|\nu - \widetilde{\nu}\|_{TV,h} \leq \|\nu\|_{TV,h} + \|\widetilde{\nu}\|_{TV,h} = \langle h, \nu\rangle + \langle h, \widetilde{\nu}\rangle$ yields

$$\begin{aligned} W_1(Q_t(\mu, \cdot), Q_t(\widetilde{\mu}, \cdot)) &\leq \int_{M_h(E)\times M_h(E)} \|\nu - \widetilde{\nu}\|_{TV,h} Q_t(\mu, d\nu) Q_t(\widetilde{\mu}, d\widetilde{\nu}) \\ &\leq \int_{M_h(E)} \langle h, \nu\rangle Q_t(\mu, d\nu) + \int_{M_h(E)} \langle h, \nu\rangle Q_t(\widetilde{\mu}, d\nu) \\ &= \langle R_t h, \mu\rangle + \langle R_t h, \widetilde{\mu}\rangle, \end{aligned}$$

where the last equality follows from Proposition 3.2 with $f = h$. This proves the desired estimate in the Wasserstein-1-distance. Since convergence in W_1 also gives convergence of first moments, we conclude that also Eq. 5.3 holds. □

In our second main result for this section, we prove the convergence towards the invariant measure under the weaker moment assumption (4.4).

Theorem 5.3 *Suppose that the (ξ, ϕ, ψ)-superprocess is exponentially subcritical and that H_2 satisfies* (4.4)*. Then there exists a constant $C > 0$ such that the unique invariant probability measure π satisfies*

$$d_{\mathcal{L}}(P_t(\mu, \cdot), \pi) \le Ce^{-\delta t}\left(1 + \log(1 + \langle h, \mu\rangle)\right). \tag{5.4}$$

and

$$d_{\mathcal{L}}(P_t^*\rho, \pi) \le Ce^{-\delta t}\left(1+\int_{M_h(E)} \log\left(1+\langle h, \mu\rangle\right)\pi(d\mu)+\int_{M_h(E)} \log\left(1+\langle h, \mu\rangle\right)\rho(d\mu)\right)$$

for all $t \ge 0$ and $\rho \in \mathcal{P}(M_h(E))$ for which the right-hand side is finite, and δ is given by Eq. 3.6.

Proof Let $C > 0$ be a generic constant which may vary from line to line. Take $f \in B_h(E)^+$, $t \ge 0$ and $\mu \in \mathcal{P}(M_h(E))$. Then

$$\begin{aligned} |\mathcal{L}_{P_t(\mu,\cdot)}(f) - \mathcal{L}_\pi(f)| &= \left|\exp\left(-\langle V_t f, \mu\rangle - \int_0^t \psi(V_s f)ds\right) - \exp\left(-\int_0^\infty \psi(V_s f)ds\right)\right| \\ &\le \exp\left(-\int_0^t \psi(V_s f)ds\right)\left|e^{-\langle V_t f,\mu\rangle} - 1\right| \\ &\quad + \left|\exp\left(-\int_0^t \psi(V_s f)ds\right) - \exp\left(-\int_0^\infty \psi(V_s f)ds\right)\right| \\ &= I_1 + I_2. \end{aligned}$$

Using $|e^{-a} - e^{-b}| \le 1 \wedge |a - b|$, $a, b \ge 0$, and then $1 \wedge a \le C\log(1 + a)$, gives

$$I_1 \le 1 \wedge \langle V_t f, \mu\rangle \le C \log\left(1 + \langle V_t f, \mu\rangle\right) \le C\|f\|_h e^{-\delta t}\left(1 + \log(1 + \langle h, \mu\rangle)\right),$$

where we have used Eq. 4.5 and $V_t f \le R_t f$. Using the estimate after Eq. 4.5 we obtain

$$I_2 \le \int_t^\infty \psi(V_s f)ds \le C\|f\|_h \int_t^\infty e^{-\delta s}ds \le C\delta^{-1}\|f\|_h e^{-\delta t}.$$

Combining both estimates gives Eq. 5.4. Let $\rho \in \mathcal{P}(M_h(E))$ and let H be any coupling of (ρ, π). Using Lemma A.4 gives

$$\begin{aligned} d_{\mathcal{L}}(P_t^*\rho, \pi) &= d_{\mathcal{L}}(P_t^*\rho, P_t^*\pi) \\ &\le \int_{M_h(E)\times M_h(E)} d_{\mathcal{L}}(P_t(\mu, \cdot), P_t(\widetilde{\mu}, \cdot))H(d\mu, d\widetilde{\mu}) \\ &\le \int_{M_h(E)\times M_h(E)} \left(d_{\mathcal{L}}(P_t(\mu, \cdot), \pi) + d_{\mathcal{L}}(\pi, P_t(\widetilde{\mu}, \cdot))\right) H(d\mu, d\widetilde{\mu}) \\ &\le \int_{M_h(E)\times M_h(E)} Ce^{-\delta t}\left(1 + \log(1+\langle h, \mu\rangle)+\log(1+\langle h, \widetilde{\mu}\rangle)\right) H(d\mu, d\widetilde{\mu}) \\ &= Ce^{-\delta t}\left(1+\int_{M_h(E)} \log(1+\langle h, \mu\rangle)\rho(d\mu)+\int_{M_h(E)} \log(1+\langle h, \mu\rangle)\pi(d\mu)\right), \end{aligned}$$

where the third inequality follows from Eq. 5.4. This completes the proof of Theorem 5.3. □

In the particular case $E = \{1, \ldots, d\}$ based on the use of Stochastic Differential Equations some similar (and essentially stronger) estimates have been recently obtained in [16]. Unfortunately the techniques developed there do not directly apply in this case. Our proof is strongly based on the representation of the Laplace transforms (2.8) combined with some properties of the Wasserstein-1-distance.

6 Examples

6.1 Super-Lévy Processes

Let $E = \mathbb{R}^d$ and denote by $C_0(\mathbb{R}^d)$ the Banach space of continuous functions vanishing at infinity. Let ξ be a Lévy process with generator $(A, D(A))$ on $C_0(\mathbb{R}^d)$ and transition semigroup

$$p_t^{\xi} f(x) = \mathbb{E}_x[f(\xi_t)], \qquad f \in C_0(\mathbb{R}^d).$$

Let ϕ be the branching mechanism given by

$$\phi(x, f) = bf(x) + cf(x)^2 + \int_0^{\infty} \left(e^{-zf(x)} - 1 + f(x)z \right) m(x, dz),$$

where $b \in \mathbb{R}$, $c \geq 0$ and $\sup_{x \in \mathbb{R}^d} \int_0^{\infty} z \wedge z^2 m(x, dz) < \infty$. For vanishing immigration (i.e. $\psi \equiv 0$) and in dimension $d = 1$ the corresponding branching process was constructed in [19]. Below we introduce immigration of mass into the system. Let ρ be a Borel probability measure on $\mathbb{R}^d$ and consider the immigration mechanism

$$\psi(f) = \int_0^{\infty} \left(1 - e^{-z\langle f, \rho \rangle} \right) G(dz),$$

where G is a Borel measure on $(0, \infty)$ such that $\int_0^{\infty} (1 \wedge z) G(dz) < \infty$.

This model describes a system of particles as follows. New mass immigrates according to a Poisson random measure with distribution ρ and intensity described by the measure G. After mass is placed inside $\mathrm{supp}(\rho) \subset \mathbb{R}^d$, it evolves according to the prescribed Lévy process ξ performing randomly some branching described by the mechanism ϕ. Below we briefly explain how the general results of this work can be applied to this case.

Suppose that there exists a strictly positive, continuous function $h \in C_0(\mathbb{R}^d)$ which belongs to the domain of the weak generator and satisfies $Ah \leq \alpha h$ for some constant $\alpha \geq 0$. Then conditions (A1) – (A3) are satisfied. Since ϕ is local, we have $\gamma(x, dy) = 0$ and hence $r_t(x, dy) = e^{-bt} p_t^{\xi}(x, dy)$ which yields

$$\int_E h(y) r_t(x, dy) = e^{-bt} \int_E h(y) p_t^{\xi}(x, dy) \leq e^{(\alpha - b)t} h(x).$$

This shows that the super-Lévy process is exponentially subcritical provided that $b > \alpha$. In particular, under the suitable integrability condition

$$\int_1^{\infty} \log(z) G(dz) < \infty,$$

such process has a unique invariant probability measure, see Section 4.

6.2 Infinite-type Continuous-State Branching Processes with Immigration

Let E be a countable (finite or infinite) discrete set equipped with the discrete topology. The reader may think about the particular cases $E = \mathbb{Z}^d$ and $E = \mathbb{N}$. Let $h = (h(x))_{x \in E}$

be a strictly positive sequence of elements indexed by E. Define the weighted space of summable sequences

$$\ell_h^1 = \left\{ \mu = (\mu(x))_{x \in E} \mid \sum_{x \in E} |\mu(x)| h(x) < \infty \right\}$$

and the weighted space of bounded sequences

$$\ell_h^\infty = \left\{ f = (f(x))_{x \in E} \mid \sup_{x \in E} \frac{|f(x)|}{h(x)} < \infty \right\}.$$

The corresponding pairing is given by

$$\langle f, \mu \rangle = \sum_{x \in E} f(x)\mu(x), \qquad f \in \ell_h^\infty, \ \ \mu \in \ell_h^1.$$

Let $(\ell_h^1)^+$ and $(\ell_h^\infty)^+$ denote the corresponding cones of nonnegative sequences. Below we consider a particle system with sites indexed by E and noncompact spins in $\mathbb{R}_+$, i.e. the state space is $(\ell_h^1)^+ \subset \mathbb{R}_+^E$. Such particle system is supposed to be described by the following objects:

(i) Let ξ be a conservative Markov chain on E whose generator is given by

$$(Af)(x) = \sum_{y \in E} (f(y) - f(x)) q(x, y), \qquad f \in B_h(E),$$

where $q(x, y) \geq 0$ is such that

$$\sup_{x \in E} \sum_{y \in E} q(x, y) < \infty. \tag{6.1}$$

Moreover there exists a constant $C > 0$ satisfying

$$\sum_{y \in E} h(y) q(x, y) \leq C h(x) + h(x) \sum_{y \in E} q(x, y), \qquad x \in E. \tag{6.2}$$

(ii) $(b(x))_{x \in E}$ is a bounded sequence.

(iii) $(c(x))_{x \in E}$ is a nonnegative sequence such that $(h(x)c(x))_{x \in E}$ is bounded.

(iv) $(\eta(x, y))_{x, y \in E}$ is an infinite matrix with nonnegative entries such that

$$\sum_{y \in E} h(y) \eta(x, y) \leq C h(x), \qquad x \in E$$

for some constant $C > 0$.

(v) $H_1(x, d\eta)$ is a collection of σ-finite measures on $(\ell_h^1)^+ \backslash \{0\}$ indexed by E and satisfying

$$\int_{(\ell_h^1)^+ \backslash \{0\}} \Big(\langle h, \nu \rangle \wedge \langle h, \nu \rangle^2 + \langle h, \nu_x \rangle \Big) H_1(x, d\nu) \leq C h(x), \qquad x \in E,$$

for some constant $C > 0$.

(vi) $(\beta(x))_{x \in E} \in (\ell_h^1)^+$ is a nonnegative sequence.

(vii) H_2 is a Borel measure on $(\ell_h^1)^+ \backslash \{0\}$ satisfying

$$\int_{(\ell_h^1)^+ \backslash \{0\}} 1 \wedge \langle h, \nu \rangle H_2(d\nu) < \infty.$$

For $x \in E$ and $(f(x))_{x\in E} \in (\ell_h^\infty)^+$ we introduce the branching mechanism

$$\phi(x,f) = c(x)f(x)^2 + b(x)f(x) - \sum_{y\in E} f(y)\eta(x,y) + \int_{(\ell_h^1)^+\setminus\{0\}} \left(e^{-\langle f,\nu\rangle} - 1 + f(x)\nu(x)\right) H_1(x,d\nu)$$

and immigration mechanism

$$\psi(f) = \sum_{x\in E} f(x)\beta(x) + \int_{(\ell_h^1)^+\setminus\{0\}} \left(1 - e^{-\langle f,\nu\rangle}\right) H_2(d\nu). \tag{6.3}$$

Theorem 6.1 *Suppose that conditions (i) – (vii) are satisfied. Then*

(a) For each $f \in (\ell_h^\infty)^+$ there exists a unique locally bounded solution $(v_t(x))_{x\in E}$ in $(\ell_h^\infty)^+$ to

$$\frac{dv_t(x)}{dt} = Av_t(x) - \phi(x,v_t), \qquad v_0(x) = f(x), \quad x \in E. \tag{6.4}$$

(b) There exists a unique Markov kernel $P_t(\eta, d\mu)$ whose Laplace transform is given by

$$\int_{(\ell_h^1)^+} e^{-\langle f,\mu\rangle} P_t(\eta,d\mu) = \exp\left(-\langle V_t f, \eta\rangle - \int_0^t \psi(V_s f)ds\right)$$

where $f \in (\ell_h^\infty)^+$ and $V_t f(x) = v_t(x)$.

(c) Assume that there exists a constant $\delta > 0$ satisfying

$$\sum_{y\in E} h(y)\left(q(x,y) + \gamma(x,y)\right) \le \left(b(x) + \sum_{y\in E} q(x,y)\right) h(x) - \delta h(x) \tag{6.5}$$

for each $x \in E$, where

$$\gamma(x,y) = \eta(x,y) + \int_{(\ell_h^1)^+\setminus\{0\}} \mathbb{1}_{\{x\neq y\}} \nu(y) H_1(x,d\nu).$$

Then the corresponding superprocess is exponentially subcritical.

Proof Note that $M_h(E)$ may be identified with $(\ell_h^1)^+$ by $\mu \longmapsto (\mu(\{x\}))_{x\in E} \equiv (\mu(x))_{x\in E}$. Similarly we can identify $B_h(E)$ with ℓ_h^∞ by $f \longmapsto (f(x))_{x\in E}$. Condition (6.1) implies that A is a bounded operator on $B_{\mathbb{1}}(E)$ and by Eq. 6.2 it is also bounded on $B_h(E)$. Hence condition (A1) is satisfied. Conditions (A2) and (A3) are immediate consequences of (ii) – (vii) so that assertions (a) and (b) follow from Theorem 2.3. Let us prove assertion (c). Observe that the right-hand side in Eq. 6.4 satisfies

$$Av_t(x) - \phi(x,v_t) = -\widetilde{\phi}(x,v_t),$$

where $\widetilde{\phi}(x,f)$ is a new branching mechanism given by

$$\widetilde{\phi}(x,f) = \widetilde{b}(x)f(x) + c(x)f(x)^2 - \sum_{y\in E} f(y)\widetilde{\gamma}(x,y) + \int_{(\ell_h^1)^+\setminus\{0\}} \left(e^{-\langle f,\nu\rangle} - 1 + \langle f,\nu\rangle\right) H_1(x,d\nu)$$

with

$$\widetilde{b}(x) = b(x) + \sum_{y \in E} q(x, y), \qquad \widetilde{\gamma}(x, y) = q(x, y) + \gamma(x, y).$$

Hence $(v_t)_{t \geq 0}$ given by Eq. 6.4 is also the unique solution to

$$\frac{dv_t(x)}{dt} = -\widetilde{\phi}(x, v_t), \qquad v_0(x) = f(x).$$

Using this new representation (which has no spatial motion) we first observe that it still satisfies conditions (A1) – (A3) for the particular choice $\xi_t = \xi_0$, $t \geq 0$. We proceed to compute the semigroup $(R_t)_{t \geq 0}$. Since $\xi_t = \xi_0$, we have $p_t^{\xi} f(x) = f(x)$ and hence in view of Eq. 3.1 the semigroup $(R_t)_{t \geq 0}$ is the unique solution to

$$R_t h(x) = h(x) + \int_0^t R_s Bh(x) ds,$$

i.e. $R_t = e^{tB}$, where B is a bounded linear operator on $B_h(E)$ given by

$$Bf(x) = \sum_{y \in E} f(y)\widetilde{\gamma}(x, y) - \widetilde{b}(x) f(x) = Af(x) + \sum_{y \in E} f(y)\gamma(x, y) - b(x) f(x).$$

By Eq. 6.5 one has $Bh(x) \leq -\delta h(x)$ which shows that $R_t h(x) \leq e^{-\delta t} h(x)$ and hence proves the assertion in view of Proposition 3.9. □

Recently in [30] the authors have studied local and global extincition for the particular choice $E = \mathbb{N}$, $h(x) \equiv 1$ with branching mechanism

$$\begin{aligned} \phi(x, f) &= b(x) f(x) + c(x) f(x)^2 + \textstyle\int_0^\infty \left(e^{-zf(x)} - 1 + zf(x)\right) G_0(x, dz) \\ &\quad -g(x) \left(d(x)\langle f, \pi_x \rangle + \textstyle\int_0^\infty \left(1 - e^{-z\langle f, \pi_x \rangle}\right) G_1(x, dz)\right), \end{aligned} \tag{6.6}$$

immigration mechanism $\psi \equiv 0$ and trivial spatial motion, i.e. $q(x, y) \equiv 0$. Here $c(x), g(x), d(x) \geq 0$ and $b(x) \in \mathbb{R}$ are bounded sequences and $(\pi_x)_{x \in \mathbb{N}}$ is a family of probability distributions on $\mathbb{N}$ with $\pi_x(\{x\}) = 0$ for all $x \in \mathbb{N}$, and $z \wedge z^2 G_0(x, dz)$ and $zG_1(x, dz)$ are bounded kernels from $\mathbb{N}$ to $(0, \infty)$. Note that the first part in ϕ corresponds to local branching while the second part is nonlocal branching which couples different lattice points in a nontrivial way. Below we introduce immigration to this model and show that it satisfies conditions (i) – (vii).

Corollary 6.2 *Let* $E = \mathbb{N}$, $h(x) \equiv 1$, ϕ *be given by* Eq. 6.6 *and* ψ *by* Eq. 6.3. *Then conditions (i) – (vii) are satisfied. If there exists* $\delta > 0$ *such that*

$$g(x) \left(d(x) + \int_0^\infty zG_1(x, dz) \right) \leq b(x) - \delta, \qquad x \in E,$$

then Eq. 6.5 *is satisfied and the corresponding superprocess is exponentially subcritical.*

Proof Let $\eta(x, dy) = g(x)d(x)\pi_x(dy)$ and set

$$H_1(x, d\nu) = \int_0^\infty \delta_{z\delta_x}(d\nu) G_0(x, dz) + \int_0^\infty \delta_{z\pi_x}(d\nu) g(x) G_1(x, dz),$$

then it is easily seen that conditions (i) – (vii) are satisfied. The second assertion follows by direct computation. □

Appendix A

A.1 Proof of Theorem 2.3

Proof Assertion (a) is a particular case of [33, Theorem 6.3]. For later use we sketch the most important step in the proof. Define a new transition semigroup $\widetilde{p}_t^{\xi}(x, dy) = e^{-\alpha t}\frac{h(y)}{h(x)}p_t^{\xi}(x, dy)$ and branching mechanism $\widetilde{\phi}(x, f) = h(x)^{-1}\phi(x, hf) - \alpha f(x)$. Then it was shown that there exists a unique locally bounded solution $u_t(x, f)$ to

$$u_t(x, f) = \widetilde{p}_t^{\xi} f(x) - \int_0^t \int_E \widetilde{\phi}(y, u_{t-s})\widetilde{p}_s^{\xi}(x, dy)ds, \qquad f \in B_{\mathbb{1}}(E)^+.$$

This solution defines a cumulant semigroup $U_t f(x) = u_t(x, f)$, i.e.

$$U_t f(x) = \int_E f(y)\widetilde{\lambda}_t(x, dy) + \int_{M(E)^\circ} \left(1 - e^{-\langle f,\nu\rangle}\right) \widetilde{L}_t(x, d\nu), \tag{A.7}$$

where $\widetilde{\lambda}_t$ and $\widetilde{L}_t$ are bounded kernels for each $t \geq 0$. Finally it was shown that

$$V_t f(x) := h(x)U_t(h^{-1} f)(x), \qquad f \in B_h(E)^+ \tag{A.8}$$

is the desired solution to Eq. 2.6, i.e. assertion (a) is proved. By uniqueness we find that $(V_t)_{t\geq 0}$ is a nonlinear semigroup. Invoking (A.7) and Eq. A.8 we find that

$$\lambda_t(x, dy) = \frac{h(x)}{h(y)}\widetilde{\lambda}_t(x, dy), \qquad L_t(x, d\nu) = h(x)\widetilde{L}(x, \cdot) \circ I^{-1}(d\nu),$$

where $I : M_{\mathbb{1}}(E)^\circ \longrightarrow M_h(E)^\circ$, $I(\nu) = h^{-1}\nu$. This proves assertion (b). Assertion (c) can be obtained as follows. Define $H_2^{(n)}(d\nu) = \mathbb{1}_{\{\langle h,\nu\rangle<n\}}H_2(d\nu)$ and let ψ_n be given by Eq. 2.4 with H_2 replaced by $H_2^{(n)}$. By [33, Proposition 9.17] there exists a unique Markov kernel $P_t^{(n)}(\mu, d\nu)$ satisfying (2.8) with ψ replaced by ψ_n. Taking the limit $n \to \infty$ and using Proposition 2.1 proves the existence of $P_t(\mu, d\nu)$ satisfying (2.8). Since $V_{s+t} = V_s V_t$, it is not difficult to see that $P_t(\mu, d\nu)$ is a Markov kernel. □

A.2 Proof of Lemma 3.1

Proof Let $(U_t)_{t\geq 0}$ be the cumulant semigroup given by Eq. A.7 and let $(\widetilde{R}_t)_{t\geq 0}$ be the semigroup on $B_{\mathbb{1}}(E)$ obtained from

$$\widetilde{R}_t f(x) = \int_E f(y)\widetilde{p}_t^{\xi}(x, dy) + \int_0^t \int_E \left(\widetilde{\Gamma}\widetilde{R}_{t-s} f(y) - \widetilde{b}(y)\widetilde{R}_{t-s} f(y)\right) \widetilde{p}_s^{\xi}(x, dy)ds,$$

where $\widetilde{p}_t^{\xi}(x, dy) = e^{-\alpha t}\frac{h(y)}{h(x)}p_t^{\xi}(x, dy)$, $\widetilde{b}(x) = b(x) - \alpha$ and

$$\widetilde{\Gamma} f(x) = \int_E f(y)\widetilde{\gamma}(x, dy), \quad \widetilde{\gamma}(x, dy) = \frac{h(y)}{h(x)}\gamma(x, dy).$$

It was shown in [33, Proposition 2.18 and 2.24] that $U_t f \leq \widetilde{R}_t f$ and $\frac{1}{\varepsilon}U_t(\varepsilon f)(x) \nearrow \widetilde{R}_t f(x)$ as $\varepsilon \searrow 0$ for each $f \in \mathbb{B}_{\mathbb{1}}(E)^+$.

Observe that $h(x)\widetilde{R}_t(h^{-1} f)(x)$ defines a semigroup of bounded linear operators on $B_h(E)^+$ which satisfies (3.1). Since the operator $Bf(x) = \Gamma f(x) - b(x)f(x)$ is a bounded linear operator on $B_h(E)$, it follows that Eq. 3.1 has a unique solution, i.e. we get $R_t f(x) = h(x)\widetilde{R}_t(h^{-1} f)(x)$. Using representation (2.7) we easily find that

$$\frac{1}{\varepsilon}V_t(\varepsilon f)(x) = h(x)\frac{1}{\varepsilon}U_t(\varepsilon h^{-1} f)(x) \nearrow h(x)\widetilde{R}_t(h^{-1} f)(x) = R_t f(x).$$

Hence using Eq. A.8 and this limit one finds Eq. 3.3. □

A.3 Proof of Proposition 3.2

Fix $\mu \in M_h(E)$ and let $f \in B_h(E)^+$. By linearity and definition of the adjoint semigroup $(R_t^*)_{t\geq 0}$ it suffices to prove for each $t \geq 0$

$$\int_{M_h(E)} \langle f, \nu\rangle P_t(\mu, d\nu) = \langle R_t f, \mu\rangle + \int_0^t \left(\langle R_s f, \beta\rangle + \int_{M_h(E)^\circ} \langle R_s f, \nu\rangle H_2(d\nu) \right) ds.$$

We start with an auxiliary result.

Lemma A.1 *For each $f \in B_h(E)^+$ and $t \geq 0$ one has*

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t \psi(V_s(\varepsilon f)) ds = \int_0^t \left(\langle R_s f, \beta\rangle + \int_{M_h(E)^\circ} \langle R_s f, \nu\rangle H_2(d\nu) \right) ds.$$

Proof Write

$$\frac{1}{\varepsilon} \int_0^t \psi(V_s(\varepsilon f)) ds = \int_0^t \int_E \frac{V_s(\varepsilon f)(x)}{\varepsilon} \beta(dx) ds + \int_0^t \int_{M_h(E)^\circ} \frac{1 - e^{-\langle V_s(\varepsilon f), \nu\rangle}}{\varepsilon} H_2(d\nu) ds.$$

Next using $\frac{V_s(\varepsilon f)}{\varepsilon} \nearrow R_s f$ pointwise and

$$\lim_{\varepsilon \searrow 0} \frac{1 - e^{-\langle V_s(\varepsilon f), \nu\rangle}}{\varepsilon} = \langle R_s f, \nu\rangle, \qquad \nu \in M_h(E)^\circ$$

with

$$\frac{1 - e^{-\langle V_s(\varepsilon f), \nu\rangle}}{\varepsilon} \leq \frac{1}{\varepsilon} \langle V_s(\varepsilon f), \nu\rangle \leq \langle R_s f, \nu\rangle$$

the assertion follows by dominated convergence. □

Based on this observation we complete the proof of Proposition 3.2.

Proof of Proposition 3.2 Fix $f \in B_h(E)^+$ and take $\varepsilon > 0$. Applying first monotone convergence and then Eq. 2.8 gives

$$\begin{aligned}
\int_{M_h(E)} \langle f, \nu\rangle P_t(\mu, d\nu) &= \lim_{\varepsilon \searrow 0} \int_{M_h(E)} \frac{1 - e^{-\langle \varepsilon f, \nu\rangle}}{\varepsilon} P_t(\mu, d\nu) \\
&= \lim_{\varepsilon \searrow 0} \frac{1 - \exp\left(-\langle V_t(\varepsilon f), \mu\rangle - \int_0^t \psi(V_s(\varepsilon f)) ds \right)}{\varepsilon} \\
&= \lim_{\varepsilon \searrow 0} \left(\int_E \frac{V_t(\varepsilon f)(x)}{\varepsilon} \mu(dx) + \frac{1}{\varepsilon} \int_0^t \psi(V_s(\varepsilon f)) ds \right) \\
&= \langle R_t f, \mu\rangle + \int_0^t \left(\langle R_s f, \beta\rangle + \int_{M_h(E)^\circ} \langle R_s f, \nu\rangle H_2(d\nu) \right) ds < \infty.
\end{aligned}$$

This proves the assertion. □

A.4 Some Results on Distances on $\mathcal{P}(M_h(E))$

Lemma A.2 *Let $\rho, \widetilde{\rho} \in \mathcal{P}(M_h(E))$, $H \in \mathcal{H}(\rho, \widetilde{\rho})$, and recall that $P_t(\mu, d\nu)$ denotes the transition kernel given by Theorem 2.3 while P_t^* was defined in* Eq. 5.1. *Then*

$$d_{\mathcal{L}}(P_t^* \rho, P_t^* \widetilde{\rho}) \leq \int_{M_h(E)\times M_h(E)} d_{\mathcal{L}}(P_t(\mu, \cdot), P_t(\widetilde{\mu}, \cdot)) H(d\mu, d\widetilde{\mu}).$$

Proof Take $f \in B_h(E)^+$. Using the definition of P_t^* and the fact that H is a coupling of $(\rho, \widetilde{\rho})$ gives

$$\begin{aligned}
&|\mathcal{L}_{P_t^*\rho}(f) - \mathcal{L}_{P_t^*\widetilde{\rho}}(f)| \\
&= \left| \int_{M_h(E)} \int_{M_h(E)\times M_h(E)} e^{-\langle f, \nu \rangle} \left(P_t(\mu, d\nu) - P_t(\widetilde{\mu}, d\nu)\right) H(d\mu, d\widetilde{\mu}) \right| \\
&\leq \int_{M_h(E)\times M_h(E)} \left| \mathcal{L}_{P_t(\mu,\cdot)}(f) - \mathcal{L}_{P_t(\widetilde{\mu},\cdot)}(f) \right| H(d\mu, d\widetilde{\mu}) \\
&\leq \|f\|_h \int_{M_h(E)\times M_h(E)} d_{\mathcal{L}}(P_t(\mu, \cdot), P_t(\widetilde{\mu}, \cdot)) H(d\mu, d\widetilde{\mu}).
\end{aligned}$$

Since f was arbitrary, the assertion is proved. □

Lemma A.3 *Let $\rho, \widetilde{\rho}, g \in \mathcal{P}(M_h(E))$. Then $W_1(\rho * g, \widetilde{\rho} * g) \leq W_1(\rho, \widetilde{\rho})$.*

Proof For $F : M_h(E) \longrightarrow \mathbb{R}$ define $\|F\|_{\text{Lip}} = \sup_{\mu \neq \widetilde{\mu}} \frac{|F(\mu) - F(\widetilde{\mu})|}{\|\mu - \widetilde{\mu}\|_{TV,h}}$. Using the Kantorovich-Duality we obtain

$$\begin{aligned}
W_1(\rho * g, \widetilde{\rho} * g) &= \sup_{\|F\|_{\text{Lip}} \leq 1} \left| \int_{M_h(E)} F(\mu)(\rho * g)(d\mu) - \int_{M_h(E)} F(\mu)(\widetilde{\rho} * g)(d\mu) \right| \\
&= \sup_{\|F\|_{\text{Lip}} \leq 1} \left| \int_{M_h(E)} F_g(\mu)\rho(d\mu) - \int_{M_h(E)} F_g(\mu)\widetilde{\rho}(d\mu) \right| \\
&\leq \sup_{\|F\|_{\text{Lip}} \leq 1} \left| \int_{M_h(E)} F(\mu)\rho(d\mu) - \int_{M_h(E)} F(\mu)\widetilde{\rho}(d\mu) \right| = W_1(\rho, \widetilde{\rho}),
\end{aligned}$$

where we used that $F_g(\mu) = \int_{M_h(E)} F(\mu + \widetilde{\mu}) g(d\widetilde{\mu})$ satisfies $\|F_g\|_{\text{Lip}} \leq 1$. □

Acknowledgements The author would like to thank the referees for their carefull reading of the manuscript and, in particular, for pointing out several inaccuracies which lead to a great improvement of this work. The author would also like to thank one of the referees for the references [12, 18, 21].

Funding Open Access funding provided by the IReL Consortium.

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