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On uniqueness and stability for the Boltzmann–Enskog equation

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Abstract. The time-evolution of a moderately dense gas in a vacuum is described in classical mechanics by a particle density function obtained from the Boltzmann–Enskog equation. Based on a McKean–Vlasov equation with jumps, the associated stochastic process was recently constructed by modified Picard iterations with the mean-field interactions, and more generally, by a system of interacting particles. By the introduction of a shifted distance that exactly compensates for the free transport term that accrues in the spatially inhomogeneous setting, we prove in this work an inequality on the Wasserstein distance for any two measure-valued solutions to the Boltzmann–Enskog equation. As a particular consequence, we find sufficient conditions for the uniqueness and continuous-dependence on initial data for solutions to the Boltzmann–Enskog equation applicable to hard and soft potentials without angular cut-off.

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1. Introduction

1.1. The Boltzmann–Enskog model

In the classical description of a moderately dense gas in a vacuum, each particle is completely described by its position $r \in \mathbb{R}^d$ and its velocity $v \in \mathbb{R}^d$, where $d \geq 3$. Moreover, the particles are assumed to be indistinguishable and with equal mass. Any particle (r, v) moves with constant speed v until it performs a collision with another particle (q, u). Denote by $v^\star, u^\star$ the resulting velocities after collision. We suppose that collisions are elastic, as a consequence conservation of momentum and kinetic energy hold, i.e.

$$u + v = u^\star + v^\star$$
$$|u|^2 + |v|^2 = |u^\star|^2 + |v^\star|^2.$$

A commonly used parameterization of the deflected velocities $v^\star, u^\star$ is given by the unit vector $n = \frac{v^\star - v}{|v^\star - v|}$ via

$$\begin{cases} v^\star &= v + (u - v, n)n \\ u^\star &= u - (u - v, n)n \end{cases}, n \in S^{d-1}, \tag{1.1}$$

where $(\cdot,\cdot)$ denotes the euclidean product in $\mathbb{R}^d$. Note that, for fixed $n \in S^{d-1}$, the change of variables $(v, u) \longmapsto (v^\star, u^\star)$ is an involutive transformation with Jacobian equal to 1.

Let $f_0(r, v) \geq 0$ be the particle density function of the gas at initial time $t = 0$. The time evolution $f_t = f_t(r, v)$ is then obtained from the Boltzmann–Enskog equation

$$\frac{\partial f_t}{\partial t} + v \cdot (\nabla_r f_t) = \mathcal{Q}(f_t, f_t), \quad f_t|_{t=0} = f_0, \quad t > 0. \tag{1.2}$$

Here $\mathcal{Q}$ is a non-local, nonlinear collision integral operator given by

$$\mathcal{Q}(f_t, f_t)(r, v) = \int_{\mathbb{R}^{2d}} \int_{S^{d-1}} (f_t(r, v^\star) f_t(q, u^\star) - f_t(r, v) f_t(q, u)) \\ \beta(r - q) B(|v - u|, n) dn du dq, \tag{1.3}$$

where dn denotes the Lebesgue surface measure on the sphere S^{d-1} and $B(|v - u|, n) \geq 0$ the collision kernel so that $B(|v - u|, n)dn$ includes the effect of velocity cross-section. The particular form of $B(|v - u|, n)$ depends on the particular microscopic model one has in mind, while $\beta(r - q) \geq 0$ describes the rate at which a particle at position r performs a collision with another particle at position q. For instance, $\beta(r - q) = \delta_0(r - q)$ describes the case of local collisions governed by the classical Boltzmann equation, while the particular choice $\beta(r - q) = \delta_\rho(|r - q|)$ describes the case where particles behave like billiard balls of radius $\rho > 0$ and was studied by Rezakhanlou [21]. Following [1,16] we study in this work the case where β is a symmetric and smooth function. Applications, additional physical background and classical mathematical results are collected in the books of Cercignani [10] and Cercignani, Illner, Pulvirenti [11]. For recent review articles on this topic we refer to Villani [24] and Alexandre [2].

1.2. Examples in dimension $d = 3$

Let us briefly comment on particular examples of collision kernels $B(|v - u|, n)$ in dimension $d = 3$. Boltzmann's original model was first formulated for (true) hard spheres, i.e. $B(|v - u|, n) = (u - v, n)$. A transformation in polar coordinates to a system where the center is in $\frac{u+v}{2}$ and $e_3 = (0, 0, 1)$ is parallel to $u - v$, i.e. $|u - v|e_3 = u - v$ leads to

$$B(|v - u|, n)dn = |(v - u, n)| = |v - u| \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) d\theta d\phi, \tag{1.4}$$

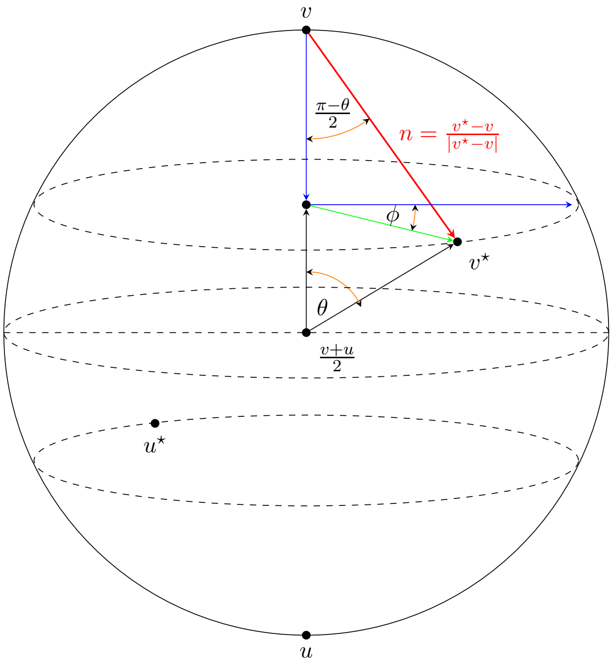

Figure 1. Parameterization of collisions

where $\theta \in (0,\pi]$ is the angle between $u - v$ and $u^\star - v^\star$ and $\phi \in (0, 2\pi]$ is the longitude angle, see Tanaka [22] or Horowitz and Karandikar [17]. This is summarized in Fig. 1.

Note that Boltzmann's original model (1.4) satisfies Grad's angular cut-off assumption, i.e.

$$\int_{S^2} B(|v-u|, n)dn < \infty.$$

A mathematically more challenging class of models which does not satisfy Grad's angular cut-off assumption is provided by *long-range interactions* given by

$$B(|v-u|, n)dn = |v-u|^\gamma b(\theta) d\theta d\xi, \tag{1.5}$$

where b is at least locally bounded on $(0,\pi]$ and

$$b(\theta) \sim \theta^{-1-\nu}, \quad \theta \to 0^+, \quad \nu \in (0,2).$$

The parameters γ and ν are related by

$$\gamma = \frac{s-5}{s-1}, \qquad \nu = \frac{2}{s-1}, \quad s > 2. \tag{1.6}$$

For long-range interactions one distinguishes between the following cases:

(i) Very soft potentials $s \in (2,3]$, $\gamma \in (-3,-1]$ and $\nu \in [1,2)$.
(ii) Soft potentials $s \in (3,5)$, $\gamma \in (-1,0)$ and $\nu \in (\frac{1}{2}, 1)$.

(iii) Maxwellian molecules $s = 5$, $\gamma = 0$ and $\nu = \frac{1}{2}$.
(iv) Hard potentials $s > 5$, $\gamma \in (0, 1)$ and $\nu \in (0, \frac{1}{2})$.

For additional details and comments we refer to [24] or [2]. Note that one has

$$\int_0^\pi b(\theta)d\theta = \infty \quad \text{but} \quad \int_0^\pi \theta^2 b(\theta)d\theta < \infty.$$

Hence $\mathcal{Q}$ is in this case a non-linear and singular integral operator with either unbounded or singular coefficients. A rigorous analysis of the corresponding Cauchy problem (1.2) is therefore a challenging mathematical task.

1.3. Literature on the Cauchy problem (1.2)

The analysis of the Cauchy problem (1.2) strongly depends on the particular choice of β, that is, on the distance of colliding particles. The classical Boltzmann equation formally corresponds to the particular choice $\beta(x-y) \equiv \delta_0(|x-y|)$ (dirac distribution at zero) where colliding particles have to be at the same position. Hence the collision integral (1.3) is local (but highly singular) in the spatial variables. Classical results on the space-inhomogeneous Boltzmann equation can be found in [2,24]. More recently, there has been some interesting progress on global solutions close to equilibrium as studied in the works of Alexandre, Morimoto, Ukai, Xu, Yang [3–6]. Note that in contrast to our work, the solutions studied in these references are, in general, not probability distributions on $\mathbb{R}^{2d}$.

Letting $\beta(x-y) \equiv \delta_\rho(|x-y|)$ provides the description of particles which can only perform elastic collisions at a fixed radius $\rho > 0$. In this case the collision integral (1.3) is less singular than in the classical Boltzmann equation. The corresponding Cauchy problem was studied e.g. by Toscani, Bellomo [23], Arkeryd [7] and Arkeryd, Cercignani [8]. Based on an interacting particle system of binary collisions, the Boltzmann-Grad limit was established for true hard spheres by Rezakhanlou [21]. Most of the results obtained in this direction are mainly applicable under Grad's angular cut-off assumption.

Finally, taking $0 \leq \beta \in C_c^1(\mathbb{R}^d)$ to be a symmetric function removes the spatial singularity in the collision integral (1.3) which allows us to use stochastic methods. This can be seen as a mollified version of either $\delta_0(|x-y|)$ or $\delta_\rho(|x-y|)$. The analysis of the corresponding Cauchy problem was initiated by Povzner [20] while propagation of chaos was studied by Cercignani [9] performing the Boltzmann-Grad limit for the corresponding BBGKY-hierarchy. First results applicable without cut-off (including the case of Maxwellian molecules) have been recently obtained in [1,16], where the construction of the corresponding stochastic process (the so-called Enskog process) was studied.

2. Statement of the result

2.1. Different parameterization of collisions

In order to study solutions to the Boltzmann–Enskog equation it is feasible to find continuity properties of the deflected velocities $v^\star, u^\star$ when the incoming velocities v, u are varied. Having in mind the case of long-range interactions

(1.5) it is also feasible to parameterize $v^\star, u^\star$ in terms of the angle θ, i.e. $n = n(v,u,\theta,\phi)$, and hence study continuity properties of $(u-v,n)n$ in u,v for fixed θ,ϕ. It was already pointed out by Tanaka that in $d=3$, $(u,v) \longmapsto (u-v,n)n$ cannot be smooth. To overcome this problem he introduced in [22] another transformation of parameters which is bijective, has Jacobian 1 and hence can be used on the right side of (1.1). Such ideas have been extended to arbitrary dimension $d \geq 3$ and are briefly summarized in this section, see [15,18]. For this purpose set $S^{d-2} = \{\xi \in \mathbb{R}^{d-1} \mid |\xi| = 1\}$ and define

$$S^{d-2}(u-v) = \{\omega \in \mathbb{R}^d \mid |u-v| = |\omega|, \quad (u-v,\omega) = 0\}.$$

The following is due to [15,22], see also [16] for this formulation.

Lemma 2.1. *Let $u,v \in \mathbb{R}^d$ with $u \neq v$ and take $n \in S^{d-1}$. Then there exist $(\theta,\xi) \in (0,\pi] \times S^{d-2}$ and a measurable bijective function $\Gamma(u-v,\cdot) : S^{d-2} \longrightarrow S^{d-2}(u-v)$, $\xi \longmapsto \Gamma(u-v,\xi)$ such that*

$$n = \sin\left(\frac{\theta}{2}\right)\frac{u-v}{|u-v|} + \cos\left(\frac{\theta}{2}\right)\frac{\Gamma(u-v,\xi)}{|u-v|}, \tag{2.1}$$

where $\theta = \theta(n) \in (0,\pi]$ be the angle between $v^\star - u^\star$ and $v-u$, i.e. it holds that $(v-u, v^\star - u^\star) = \cos(\theta)|v-u||v^\star - u^\star|$.

The representation of the vector n in (2.1) corresponds to the blue lines in Fig. 1. Inserting this into (1.1) gives after a short computation

$$\begin{cases} v^\star &= v + \alpha(v,u,\theta,\xi) \\ u^\star &= u - \alpha(v,u,\theta,\xi) \end{cases}, \tag{2.2}$$

where

$$\alpha(v,u,\theta,\xi) = \sin^2\left(\frac{\theta}{2}\right)(u-v) + \frac{\sin(\theta)}{2}\Gamma(u-v,\xi). \tag{2.3}$$

Note that (2.2) remains true also for $v=u$, if we let $\alpha(v,v,\theta,\xi)=0$, i.e. set $\Gamma(0,\xi)=0$ in (2.3). Using (2.3) one finds for all $u,v \in \mathbb{R}^d$, $\theta \in (0,\pi]$ and $\xi \in S^{d-2}$ the identity

$$|\alpha(v,u,\theta,\xi)| = |v-u|\sin\left(\frac{\theta}{2}\right). \tag{2.4}$$

From now on we work with the parameterization (2.2), where α is given by (2.3).

2.2. Some notation

Here and below we let $\langle v \rangle := (1+|v|^2)^{1/2}$ and frequently use the elementary inequalities

$$\langle v+w \rangle \leq \sqrt{2}\left(\langle v \rangle + \langle w \rangle\right) \qquad \text{and} \qquad \langle v+w \rangle \leq \sqrt{2}\langle v \rangle \langle w \rangle.$$

We denote by $K, C > 0$ generic constants which may vary from line to line. Finally, for $k \in \mathbb{N}$ we use the following function spaces

- $C^k(\mathbb{R}^{2d})$ the space of all continuous functions on $\mathbb{R}^{2d}$ which are k-times continuously differentiable.

- $C_b^k(\mathbb{R}^{2d})$ the space of all $f \in C^k(\mathbb{R}^{2d})$ such that f and its first k derivatives are bounded.
- $C_c^k(\mathbb{R}^{2d})$ the space of all $f \in C^k(\mathbb{R}^{2d})$ such that f has compact support.
- $\mathrm{Lip}(\mathbb{R}^{2d})$ the space of all globally Lipschitz continuous functions.

Denote by $\mathcal{P}(\mathbb{R}^d)$ the space of probability measures and let

$$\langle \psi, \mu \rangle = \int_{\mathbb{R}^{2d}} \psi(r,v) d\mu(r,v)$$

be the pairing between $\mu \in \mathcal{P}(\mathbb{R}^{2d})$ and an integrable function ψ.

2.3. Weak formulation for measure-solutions of the Boltzmann–Enskog equation

In this work we take any dimension $d \geq 3$ and assume that the collision kernel B is given by the velocity cross-section $\sigma \geq 0$ and a measure Q such that

$$B(|v-u|, n)dn \equiv \sigma(|v-u|)Q(d\theta)d\xi, \qquad \kappa := \int_{(0,\pi]} \theta Q(d\theta) < \infty \tag{2.5}$$

where $d\xi$ is the Lebesgue surface measure on S^{d-2} (recall (2.1)). Moreover suppose that $0 \leq \beta \in C_c^1(\mathbb{R}^d)$ is symmetric and, there exists $\gamma \in (-d, 2]$ and $c_\sigma \geq 1$ such that

$$|\sigma(|z|) - \sigma(|w|)| \leq c_\sigma ||z|^\gamma - |w|^\gamma|, \quad z, w \in \mathbb{R}^d \backslash \{0\}.$$

and

$$\sigma(|z|) \leq c_\sigma \begin{cases} |z|^\gamma, & \gamma \in (-d, 0] \\ (1+|z|^2)^{\frac{\gamma}{2}}, & \gamma \in [0,2] \end{cases}.$$

Without loss of generality we assume that β is bounded by 1.

Remark 2.2. These conditions are satisfied for $\sigma(z) = |z|^\gamma$ and also $\sigma(|z|) = (1+|z|^2)^{\frac{\gamma}{2}}$ with $\gamma \in (-d, 2]$. In particular, we cover the case of hard and soft potentials, provided $s > 3$.

Below we describe the weak formulation of the Boltzmann–Enskog equation for measures, see [1,16] for additional details. Set $\Xi = (0,\pi] \times S^{d-2}$ and, for $\psi \in C_b^1(\mathbb{R}^{2d})$, let

$$\begin{aligned} (\mathcal{A}\psi)(r,v;q,u) &= v \cdot (\nabla_r \psi)(r,v) + \sigma(|v-u|)\beta(r-q)(\mathcal{L}\psi)(r,v;u), \\ (\mathcal{L}\psi)(r,v;u) &= \int_\Xi (\psi(r, v + \alpha(v,u,\theta,\xi)) - \psi(r,v))\, Q(d\theta)d\xi. \end{aligned} \tag{2.6}$$

By (2.4) we obtain $|\alpha(v,u,\theta,\xi)| \leq \theta|v-u|$ and

$$|\psi(r, v + \alpha(v,u,\theta,\xi)) - \psi(r,v)| \leq \theta|v-u| \max_{|\zeta| \leq 2(|v|+|u|)} |\nabla_\zeta \psi(r,\zeta)|. \tag{2.7}$$

In particular, $(\mathcal{L}\psi)(r,v;u)$ is well-defined for all r, v, u and all $\psi \in C^1(\mathbb{R}^{2d})$. Moreover, if $\psi \in C_b^1(\mathbb{R}^{2d})$, then we obtain

$$|\mathcal{A}\psi(r,v;q,u)| \leq \|\nabla_r \psi\|_\infty |v| + \|\nabla_v \psi\|_\infty |v-u| \sigma(|v-u|) |S^{d-2}| \kappa. \tag{2.8}$$

Definition 2.3. Let $\mu_0 \in \mathcal{P}(\mathbb{R}^{2d})$ and fix $T > 0$. A weak solution to the Boltzmann–Enskog equation is a family $(\mu_t)_{t\in[0,T]} \subset \mathcal{P}(\mathbb{R}^{2d})$ such that

$$\begin{cases} \int_0^T \int_{\mathbb{R}^{4d}} |v-u|^{1+\gamma} \mu_t(dr,dv)\mu_t(dq,du)dt < \infty, & \text{for } \gamma \in (-d,0) \\ \int_0^T \int_{\mathbb{R}^{2d}} |v|^{1+\gamma}\mu_t(dr,dv)dt < \infty, & \text{for } \gamma \in [0,2] \end{cases} \tag{2.9}$$

and, for any $\psi \in C_b^1(\mathbb{R}^{2d})$, we have

$$\langle \psi, \mu_t \rangle = \langle \psi, \mu_0 \rangle + \int_0^t \langle \mathcal{A}\psi, \mu_s \otimes \mu_s \rangle ds, \quad t \in [0,T]. \tag{2.10}$$

Analogously we define a global weak solution to the Boltzmann–Enskog equation.

Note that one has $\mathcal{A}1 = 0$ where 1 denotes the constant function equal to one. Hence total mass is conserved and we may restrict our study of the Boltzmann–Enskog equation without loss of generality to the case of probability distributions. A construction of such solutions was recently studied in [1,16].

2.4. Stability estimates for the Boltzmann–Enskog equation

In this work we prove stability estimates for weak solutions to the Boltzmann–Enskog equation in the Wasserstein distance. In contrast to the space-homogeneous case studied in [15] the additional free transport term $v \cdot \nabla_r$ prevents us from directly applying their methods. In order to take this transport of particles into account we introduce the *shifted* Wasserstein distance

$$W_1^t(\mu,\nu) := W_1(S(-t)^*\mu, S(-t)^*\nu), \qquad t \in \mathbb{R},$$

where $S(t)$ is a one-parameter group of transformations defined by

$$S(t)\psi(r,v) = \psi(r+tv,v), \qquad (r,v) \in \mathbb{R}^{2d}, \quad t \in \mathbb{R},$$

$S(t)^*$ denotes the adjoint operator to $S(t)$ acting on measures $\mu \in \mathcal{P}(\mathbb{R}^{2d})$ and $\psi \in C_b(\mathbb{R}^{2d})$ via

$$\langle S(t)\psi, \mu \rangle = \langle \psi, S(t)^*\mu \rangle, \qquad t \in \mathbb{R}. \tag{2.11}$$

and W_1 denotes the classical Wasserstein distance, i.e.

$$W_1(\mu,\nu) := \sup_{\|\psi\|_{\mathrm{Lip}} \le 1} \langle \psi, \mu - \nu \rangle,$$

where $\mathrm{Lip}(\mathbb{R}^{2d}) = \{\psi \mid \|\psi\|_{\mathrm{Lip}} < \infty\}$ is the space of all globally Lipschitz continuous functions and

$$\|\psi\|_{\mathrm{Lip}} = \sup_{(r,v)\neq(q,u)} \frac{|\psi(r,v) - \psi(q,u)|}{|r-q| + |v-u|}.$$

In the case of hard potentials we obtain the following.

Theorem 2.4. *Suppose that $\gamma \in [0,2]$, fix $T > 0$ and $\delta > 0$. Then there exists a constant $K > 0$ such that for two given weak solutions $(\mu_t)_{t\in[0,T]}, (\nu_t)_{t\in[0,T]}$ to the Boltzmann–Enskog equation satisfying*

$$\mathcal{C}_\gamma(T, \mu+\nu, \delta) := \sup_{t\in[0,T]} \int_{\mathbb{R}^{2d}} \left(e^{\delta|v|^{1+\gamma}} + |r|^{1+\delta} \right) (\mu_t + \nu_t)(dr,dv) < \infty \tag{2.12}$$

we have, for $t \in [0,T]$,

$$W_1^t(\mu_t,\nu_t) \le W_1(\mu_0,\nu_0) + K\mathcal{C}_\gamma(T,\mu+\nu,\delta)\int_0^t W_1^s(\mu_s,\nu_s)\,(1+|\log(W_1^s(\mu_s,\nu_s))|)\,ds.$$

For soft potentials we obtain the following.

Theorem 2.5. *Suppose that* $\gamma \in (-d,0)$ *and fix* $T>0$.

(a) *If* $\gamma \in (-d,-1]$, *then there exists a constant* $K>0$ *such that for two given weak solutions* $(\mu_t)_{t\in[0,T]}, (\nu_t)_{t\in[0,T]}$ *to the Boltzmann–Enskog equation satisfying*

$$\Lambda(\mu_t+\nu_t) := \sup_{u\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} |v-u|^\gamma(\mu_t+\nu_t)(dr,dv) < \infty \tag{2.13}$$

and (2.12) *we have, for* $t\in[0,T]$,

$$W_1^t(\mu_t,\nu_t) \le W_1(\mu_0,\nu_0)\exp\left(K\int_0^t(1+\Lambda(\mu_s+\nu_s))ds\right).$$

(b) *If* $\gamma\in(-1,0)$. *Then for each* $\delta>0$ *there exists a constant* $K>0$ *such that for two given weak solutions* $(\mu_t)_{t\in[0,T]}, (\nu_t)_{t\in[0,T]}$ *to the Boltzmann–Enskog equation satisfying* (2.12) *and* (2.13) *we have, for* $t\in[0,T]$,

$$W_1^t(\mu_t,\nu_t) \le W_1(\mu_0,\nu_0) + K\mathcal{C}_\gamma(T,\mu+\nu,\delta) \cdot \sup_{s\in[0,T]}\Lambda(\mu_s+\nu_s)\int_0^t W_1^s(\mu_s,\nu_s)\,(1+|\log(W_1^s(\mu_s,\nu_s))|)\,ds.$$

Condition (2.13) stems from the necessity to compensate the singularity of $\sigma(|v-u|)$ at zero appearing in the case of soft potentials, see (1.5).

Remark 2.6. Suppose that $\gamma\in(-d,0)$ and let $\mu_t(dr,dv) = f_t(r,v)drdv$, $\nu_t(dr,dv) = g_t(r,v)drdv$. Then for each $p > \frac{d}{d+\gamma}$ there exists a constant $C(p,\gamma)>0$ such that

$$\Lambda(\mu_t+\nu_t) \le 2 + C(p,\gamma)\left[\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f_t(r,v)dr\right)^p dv\right)^{1/p} + \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} g_t(r,v)dr\right)^p dv\right)^{1/p}\right].$$

Above estimates are sufficient to imply uniqueness and stability (with respect to initial data) of weak solutions to the Boltzmann–Enskog equation. Indeed, by using a generalization of the Gronwall inequality as stated in the "Appendix" (see e.g. [12, Lemma 5.2.1, p. 89]) we obtain the following.

Theorem 2.7. *Fix* $T>0$.

(a) *Let* $(\mu_t)_{t\in[0,T]}$, $(\nu_t)_{t\in[0,T]}$ *be two weak solutions to the Boltzmann–Enskog equation. Suppose one of the following conditions is satisfied:*
   - $\gamma\in[0,2]$ *and there exists* $\delta>0$ *with*

$$\mathcal{C}_\gamma(T,\mu+\nu,\delta)<\infty.$$

- $\gamma \in (-1, 0)$ *and there exists* $\delta > 0$ *with*

$$\mathcal{C}_\gamma(T, \mu + \nu, \delta) + \sup_{t \in [0,T]} \Lambda(\mu_t + \nu_t) < \infty.$$

- $\gamma \in (-d, -1]$,

$$\sup_{t \in [0,T]} \left\{ \Lambda(\mu_t + \nu_t) + \int_{\mathbb{R}^{2d}} |v|^2 (\mu_t(dr, dv) + \nu_t(dr, dv)) \right\} < \infty.$$

If $\mu_0 = \nu_0$, *then* $\mu_t = \nu_t$ *for all* $t \in [0, T]$.

(b) *Let* $(\mu_t^{(n)})_{t \in [0,T]}$ *and* $(\mu_t)_{t \in [0,T]}$ *be weak solutions to the Boltzmann–Enskog equation. Suppose one of the following conditions is satisfied:*

- $\gamma \in [0, 2]$ *and there exists* $\delta > 0$ *with*

$$\sup_{n \in \mathbb{N}} \mathcal{C}_\gamma(T, \mu^{(n)} + \mu, \delta) < \infty.$$

- $\gamma \in (-1, 0)$ *and there exists* $\delta > 0$ *with*

$$\sup_{n \in \mathbb{N}} \left\{ \mathcal{C}_\gamma(T, \mu^{(n)} + \mu, \delta) + \sup_{t \in [0,T]} \Lambda(\mu_t^{(n)} + \mu_t) \right\} < \infty.$$

- $\gamma \in (-d, -1]$ *and*

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \left\{ \Lambda(\mu_t^{(n)} + \mu_t) + \int_{\mathbb{R}^{2d}} |v|^2 (\mu_t^{(n)}(dr, dv) + \mu_t(dr, dv)) \right\} < \infty.$$

If $W_1(\mu_0^{(n)}, \mu_0) \longrightarrow 0$ *as* $n \to \infty$, *then*

$$\lim_{n \to \infty} \sup_{t \in [0,T]} W_1^t(\mu_t^{(n)}, \mu_t) = 0.$$

Our proofs are partially inspired by the work of Fournier and Mouhot [15] where similar estimates for solutions to the space-homogeneous Boltzmann equation have been established. Other uniqueness results for the space-homogeneous Boltzmann equation are based on additional regularity assumptions for the solution, see e.g. [13,26]. However, since we work in the space-inhomogeneous setting we have to replace the classical Wasserstein distance W_1 to by a shifted distance W_1^t which compensates the free transport operator $v \cdot \nabla_r$ appearing in the definition of $\mathcal{A}$, see (2.6). For hard-potentials the authors have used in [15] Povzner inequalities to prove creation of exponential moments for solutions to the (space-homogeneous) Boltzmann equation, see also [19] and the references therein. Their proofs implicitly use the fact that any two particles may perform a collision. In contrast to that, in the space-inhomogeneous setting studied here β is compactly supported and hence only particles being close enough may perform a collision. This prevents us from proving similar results on the creation of moments for the Boltzmann–Enskog equation with hard potentials.

While this is one of the few results where uniqueness and stability estimates are obtained for the space-inhomogeneous setting, future research may be devoted to the study of suitable moment estimates in the spirit of Povzner or to the regularity of solutions to the Enskog equation. Any progress in this regard would also help to find a closed space of probability measures on which the Enskog equation is well-posed.

3. Mild formulation for the Boltzmann–Enskog equation

In order to prove the desired stability estimates for the shifted distance W_1^t, it is reasonable to use another formulation of the Boltzmann–Enskog equation which involves the semigroup $S(t)$. This is precisely the content of this section. Define for $(r,v),(q,u) \in \mathbb{R}^{2d}$ and $\psi \in \mathrm{Lip}(\mathbb{R}^{2d})$

$$(\mathcal{B}\psi)(r,v;q,u) = \sigma(|v-u|)\beta(r-q)(\mathcal{L}\psi)(r,v;u).$$

Then there exists a constant $C>0$ such that for each $\psi \in \mathrm{Lip}(\mathbb{R}^{2d})$ one has

$$|(\mathcal{B}\psi)(r,v;q,u)| \leq C|v-u|\sigma(|v-u|)\|\psi\|_{\mathrm{Lip}}. \tag{3.1}$$

The next result is crucial for estimating weak solutions to the Boltzmann–Enskog equation.

Proposition 3.1. *Fix $T>0$ and let $(\mu_t)_{t\in[0,T]} \subset \mathcal{P}(\mathbb{R}^{2d})$ satisfy*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^{2d}} |v|^2 \mu_t(dr,dv) < \infty. \tag{3.2}$$

If $(\mu_t)_{t\in[0,T]}$ is a weak solution to the Boltzmann–Enskog equation, then

$$\langle \psi, \mu_t\rangle = \langle S(t)\psi, \mu_0\rangle + \int_0^t \langle \mathcal{B}S(t-s)\psi, \mu_s \otimes \mu_s\rangle ds, \quad t \in [0,T]. \tag{3.3}$$

holds for each $\psi \in \mathrm{Lip}(\mathbb{R}^{2d})$.

Proof. Fix $\psi \in C_b^2(\mathbb{R}^{2d})$ and $t \in (0,T]$. Let us show that the function $[0,t] \ni s \longmapsto \langle S(t-s)\psi, \mu_s\rangle$ is absolutely continuous and for a.a. $s \in [0,t)$ it holds that

$$\frac{d}{ds}\langle S(t-s)\psi, \mu_s\rangle = \langle \mathcal{B}S(t-s)\psi, \mu_s \otimes \mu_s\rangle. \tag{3.4}$$

In such a case, using (3.2) and (3.1), we may integrate (3.4) over $[0,t]$ which would readily yield (3.3) for $\psi \in C_b^2(\mathbb{R}^{2d})$. If $\psi \in \mathrm{Lip}(\mathbb{R}^{2d})$ then we may find a sequence of functions $\psi_n \in C_b^2(\mathbb{R}^{2d})$ such that $\sup_{n\in\mathbb{N}} \|\psi_n\|_{\mathrm{Lip}} < \infty$ and $\psi_n \longrightarrow \psi$ pointwise. Hence passing to the limit $n \to \infty$ proves that (3.3) also holds for $\psi \in \mathrm{Lip}(\mathbb{R}^{2d})$.

Arguing in this way, it remains to prove (3.4) for each $\psi \in C_b^2(\mathbb{R}^{2d})$ and fixed $t>0$. Take $s \in [0,t)$ and let $h \in \mathbb{R}$ with $|h| \leq (t-s) \wedge 1$. Write

$$\frac{\langle S(t-(s+h))\psi, \mu_{s+h}\rangle - \langle S(t-s)\psi, \mu_s\rangle}{h}$$
$$= \left\langle \frac{S(t-(s+h))\psi - S(t-s)\psi}{h}, \mu_{s+h}\right\rangle + \frac{\langle S(t-s)\psi, \mu_{s+h}\rangle - \langle S(t-s)\psi, \mu_s\rangle}{h}.$$

Since $(\mu_t)_{t\geq 0}$ satisfies the Boltzmann–Enskog equation and $S(t-s)\psi \in C_b^2(\mathbb{R}^{2d})$ we conclude that

$$\frac{\langle S(t-s)\psi, \mu_{s+h}\rangle - \langle S(t-s)\psi, \mu_s\rangle}{h} \longrightarrow \langle \mathcal{A}S(t-s)\psi, \mu_s \otimes \mu_s\rangle, \qquad h \to 0 \tag{3.5}$$

for a.a. $s \in [0,t)$. Next we will prove that

$$\left\langle \frac{S(t-(s+h))\psi - S(t-s)\psi}{h}, \mu_{s+h} \right\rangle \longrightarrow -\int_{\mathbb{R}^{2d}} v \cdot (\nabla_r S(t-s)\psi)(r,v)\mu_s(dr,dv) \tag{3.6}$$

as $h \to 0$. Combining then (3.5), (3.6) and using the definition of $\mathcal{A}$ and $\mathcal{B}$ proves (3.4). In order to prove (3.6) we let

$$f_h(r,v) = S(t-(s+h))\psi(r,v) = \psi(r+(t-s-h)v, v)$$

and denote the corresponding pointwise derivative with respect to h at $h=0$ by

$$f_0'(r,v) = -v \cdot (\nabla_r \psi)(r+(t-s)v, v).$$

For $R > 0$ take a smooth function φ_R on $\mathbb{R}^d$ such that $\mathbb{1}_{[0,R]}(|v|) \leq \varphi_R(v) \leq \mathbb{1}_{[0,2R]}(|v|)$. Then

$$\begin{aligned}
&\left\langle \frac{S(t-(s+h))\psi - S(t-s)\psi}{h}, \mu_{s+h} \right\rangle \\
&= \left\langle \frac{f_h - f_0}{h} - f_0', \mu_{s+h} \right\rangle + \langle f_0', \mu_{s+h}\rangle \\
&= \left\langle \frac{f_h - f_0}{h} - f_0', \mu_{s+h} \right\rangle + \langle f_0'\varphi_R, \mu_{s+h} - \mu_s\rangle + \langle f_0'(1-\varphi_R), \mu_{s+h} - \mu_s\rangle + \langle f_0', \mu_s\rangle \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Using the estimate

$$\left| \frac{f_h(r,v) - f_0(r,v)}{h} - f_0'(r,v) \right| \leq \frac{|h|}{2}|v|^2 \|\psi\|_{C_b^2}$$

we obtain

$$|J_1| \leq \frac{|h|}{2}\|\psi\|_{C_b^2} \sup_{|h|\leq (t-s)\wedge 1} \int_{\mathbb{R}^{2d}} |v|^2 \mu_{s+h}(dr,dv).$$

Similarly, using $|f_0'(r,v)| \leq |v|\|\psi\|_{C_b^1}$ gives $f_0'\varphi_R \in C_b(\mathbb{R}^{2d})$ and hence we obtain

$$\lim_{h\to 0} J_2 = 0, \qquad \forall R > 0 \,\text{and}\, s > 0,$$

since $\mathbb{R}_+ \ni s \longmapsto \langle f_0'\varphi_R, \mu_s\rangle$ is continuous which essentially follows from (2.8) combined with (2.10), and a standard approximation of $C_b^1(\mathbb{R}^{2d})$ functions by $C_b(\mathbb{R}^{2d})$ functions. For J_3 we use $1 - \varphi_R(v) \leq \mathbb{1}_{(R,\infty)}(|v|)$ so that

$$|J_3| \leq \|\psi\|_{C_b^1} \int_{|v|>R} |v|(\mu_{s+h}(dr,dv) + \mu_s(dr,dv))$$

$$\leq \frac{2\|\psi\|_{C_b^1}}{R} \sup_{|h|\leq(t-s)\wedge 1} \int_{\mathbb{R}^{2d}} |v|^2 \mu_{s+h}(dr, dv).$$

Combining these estimates yields (3.6) as $h \to 0$ and letting $R \to \infty$ thus completes the proof of Proposition 3.1. □

4. The coupling inequality

Below we first provide another representation for the metric $W_1^t(\mu, \nu)$ in terms of optimal couplings. Additional details on the classical Wasserstein distance and optimal transport are given in [25]. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^{2d})$. A coupling H of (μ, ν) is a probability measure on $\mathbb{R}^{4d}$ such that its marginals are given by μ and ν, respectively. Let $\mathcal{H}(\mu, \nu)$ the space of all such couplings. Define a one-parameter family of norms on $\mathbb{R}^{2d}$ via

$$|(r, v) - (\widetilde{r}, \widetilde{v})|_t := |(r - vt) - (\widetilde{r} - \widetilde{v}t)| + |v - \widetilde{v}|, \quad t \geq 0.$$

Related to this family we define the Lipschitz norms

$$\|\psi\|_t := \sup_{(r,v)\neq(\widetilde{r},\widetilde{v})} \frac{|\psi(r, v) - \psi(\widetilde{r}, \widetilde{v})|}{|(r, v) - (\widetilde{r}, \widetilde{v})|_t}, \quad t \geq 0.$$

Note that these Lipschitz norms are all equivalent. We will use the following simple observation.

Lemma 4.1. *Let μ, ν be probability measures with finite first moments and take $t \geq 0$. Then there exists $H_t \in \mathcal{H}(\mu, \nu)$ such that*

$$\begin{aligned} W_1^t(\mu, \nu) &= \sup_{\|\psi\|_0\leq 1} \langle S(-t)\psi, \mu - \nu\rangle \\ &= \sup_{\|\psi\|_t\leq 1} \langle \psi, \mu - \nu\rangle = \int_{\mathbb{R}^{4d}} |(r, v) - (\widetilde{r}, \widetilde{v})|_t dH_t(r, v; \widetilde{r}, \widetilde{v}). \end{aligned} \tag{4.1}$$

Proof. The first equality in (4.1) follows by definition of W_1^t combined with (2.11). For the second equality in (4.1) observe that, for any ψ with $\|\psi\|_0 \leq 1$, we get $\|S(-t)\psi\|_t \leq 1$. Conversely any ψ satisfying $\|\psi\|_t \leq 1$ can be written as $\psi = S(-t)\phi$ where $\phi := S(t)\psi$ satisfies $\|\phi\|_0 \leq 1$. The last equality in (4.1) is a particular case of the Kantorovich duality for Wasserstein distances (see e.g. [25, Theorem 5.10]). □

The following is an extension of [15, Theorem 2.2] to the space-inhomogeneous case.

Proposition 4.2. *Take $T > 0$ and let $(\mu_t)_{t\in[0,T]}, (\nu_t)_{t\in[0,T]}$ be two weak solutions to the Boltzmann–Enskog equation satisfying*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^{2d}} |v|^2 (\mu_t(dr, dv) + \nu_t(dr, dv)) < \infty,$$

and

$$\int_0^T \int_{\mathbb{R}^{4d}} (|v| + |u| + |r| + |q|)\sigma(|v - u|)$$

$$(\mu_t(dq,du)\mu_t(dr,dv) + \nu_t(dq,du)\nu_t(dr,dv))\,dt < \infty. \quad (4.2)$$

For $t \in [0,T]$ let $H_t \in \mathcal{H}(\mu_t,\nu_t)$ be such that

$$W_1^t(\mu_t,\nu_t) = \int_{\mathbb{R}^{4d}} |(r,v) - (\widetilde{r},\widetilde{v})|_t dH_t(r,v;\widetilde{r},\widetilde{v}). \quad (4.3)$$

Then for all $t \in [0,T]$

$$W_1^t(\mu_t,\nu_t) \le W_1(\mu_0,\nu_0) + 2\kappa(1+T)\int_0^t \int_{\mathbb{R}^{8d}} \Psi dH_s(q,u;\widetilde{q},\widetilde{u})dH_s(r,v;\widetilde{r},\widetilde{v})ds$$

where

$$\begin{aligned}\Psi &:= (|v-u| + |\widetilde{v}-\widetilde{u}|)\,|\sigma(|v-u|)\beta(r-q) - \sigma(|\widetilde{v}-\widetilde{u}|)\beta(\widetilde{r}-\widetilde{q})| \\ &\quad + (|v-\widetilde{v}| + |u-\widetilde{u}|)\min\{\sigma(|v-u|)\beta(r-q), \sigma(|\widetilde{v}-\widetilde{u}|)\beta(\widetilde{r}-\widetilde{q})\}.\end{aligned}$$

In order to prove this result we need continuity properties for the collision integral, i.e. to compare (2.3) for different values of u, v. It was already pointed out by Tanaka that $(u,v) \longmapsto \alpha(v,u,\theta,\xi)$ cannot be smooth for any choice of (θ,ξ). Using the parameterization (2.2) Tanaka [22, Lemma 3.1] has shown that if we allow to shift the angles ξ in a suitable way, then a weaker form of continuity holds. The latter estimate is sufficient for this work. Below we recall Tanaka's result for arbitrary dimension $d \ge 3$ which is due to [15].

Lemma 4.3. [15, Lemma 3.1] *There exists a measurable map $\xi_0 : \mathbb{R}^d \times \mathbb{R}^d \times S^{d-2} \longrightarrow S^{d-2}$ such that for any $X, Y \in \mathbb{R}^d\backslash\{0\}$, the map $\xi \longmapsto \xi_0(X,Y,\xi)$ is a bijection with Jacobian 1 from S^{d-2} onto itself, and*

$$|\Gamma(X,\xi) - \Gamma(Y,\xi_0(X,Y,\xi))| \le 3|X-Y|, \qquad \xi \in S^{d-2}.$$

With this parameterization we obtain from Lemma 4.3, for all $u, v, \widetilde{u}, \widetilde{v} \in \mathbb{R}^d$, all $\theta \in [0,\pi]$ and all $\xi \in S^{d-2}$, we have the inequality

$$|\alpha(v,u,\theta,\xi) - \alpha(\widetilde{v},\widetilde{u},\theta,\xi_0(v-u,\widetilde{v}-\widetilde{u},\xi))| \le 2\theta\,(|v-\widetilde{v}| + |u-\widetilde{u}|)\,. \quad (4.4)$$

We are now prepared to prove our main coupling inequality of this section.

Proof of Proposition 4.2. Take $\psi \in \mathrm{Lip}(\mathbb{R}^{2d})$ with $\|\psi\|_0 \le 1$. By Proposition 3.1 $(\mu_t)_{t\in[0,T]}$ and $(\nu_t)_{t\in[0,T]}$ also satisfy (3.3). To shorten notation we let $\alpha = \alpha(v,u,\theta,\xi)$, $\sigma = \sigma(|v-u|)$, $\beta = \beta(r-q)$ and likewise $\widetilde{\alpha}, \widetilde{\sigma}, \widetilde{\beta}$ with $(r,v),(q,u)$ replaced by $(\widetilde{r},\widetilde{v}),(\widetilde{q},\widetilde{u})$. Moreover, let $dH_s^0 = dH_s(q,u;\widetilde{q},\widetilde{u})$ and $dH_s^1 = dH_s(r,v;\widetilde{r},\widetilde{v})$. Using (3.3), $H_s \in \mathcal{H}(\mu_s,\nu_s)$, the definition of $\mathcal{B}$ and finally $x = x \wedge y + (x-y)_+$, for $x, y \ge 0$ with $x_+ := \max\{x,0\}$, we obtain

$$\begin{aligned}&\langle S(-t)\psi, \mu_t - \nu_t\rangle - \langle \psi, \mu_0 - \nu_0\rangle \\ &\quad = \int_0^t \Big\{\langle \mathcal{B}S(-s)\psi, \mu_s \otimes \mu_s\rangle - \langle \mathcal{B}S(-s)\psi, \nu_s \otimes \nu_s\rangle\Big\}ds \\ &\quad = \int_0^t \int_{\mathbb{R}^{8d}} \Big\{(\mathcal{B}S(-s)\psi)(r,v;q,u) \\ &\qquad - (\mathcal{B}S(-s)\psi)(\widetilde{r},\widetilde{v};\widetilde{q},\widetilde{u})\Big\}dH_s(q,u,\widetilde{q},\widetilde{u})dH_s(r,v;\widetilde{r},\widetilde{v})ds.\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \int_{\mathbb{R}^{8d}\times\Xi} \Big\{ \left(S(-s)\psi(r,v+\alpha) - S(-s)\psi(r,v)\right)\sigma\beta \\
&\quad - \left(S(-s)\psi(\widetilde{r},\widetilde{v}+\widetilde{\alpha}) - S(-s)\psi(\widetilde{r},\widetilde{v})\right)\widetilde{\sigma}\widetilde{\beta}\Big\} dQ d\xi dH_s^0 dH_s^1 ds \\
&= \int_0^t \int_{\mathbb{R}^{8d}\times\Xi} \left(\sigma\beta \wedge \widetilde{\sigma}\widetilde{\beta}\right) S(-s) \Big\{ \psi(r,v+\alpha) - \psi(\widetilde{r},\widetilde{v}+\widetilde{\alpha}) \\
&\quad - \psi(r,v) + \psi(\widetilde{r},\widetilde{v})\Big\} dQ d\xi dH_s^0 dH_s^1 ds \\
&\quad + \int_0^t \int_{\mathbb{R}^{8d}\times\Xi} \left(\sigma\beta - \widetilde{\sigma}\widetilde{\beta}\right)_+ \left(S(-s)\psi(r,v+\alpha) - S(-s)\psi(r,v)\right) dQ d\xi dH_s^0 dH_s^1 ds \\
&\quad - \int_0^t \int_{\mathbb{R}^{8d}\times\Xi} \left(\widetilde{\sigma}\widetilde{\beta} - \sigma\beta\right)_+ \left(S(-s)\psi(\widetilde{r},\widetilde{v}+\widetilde{\alpha}) - S(-s)\psi(\widetilde{r},\widetilde{v})\right) dQ d\xi dH_s^0 dH_s^1 ds \\
&=: J_1 + J_2 + J_3.
\end{aligned}
$$

Note that, by (2.4), we have $|(0,\alpha)|_s \le (1+s)|\alpha| \le (1+T)\theta|v-u|$ where 0 denotes the zero vector in $\mathbb{R}^d$. Analogously we obtain $|(0,\widetilde{\alpha})|_s \le (1+T)\theta|\widetilde{v}-\widetilde{u}|$. Hence using $\|S(-s)\psi\|_s \le 1$ we get

$$
\begin{aligned}
J_2 + J_3 &\le \int_0^t \int_{\mathbb{R}^{8d}} \int_{\Xi} \left(|(0,\alpha)|_s + |(0,\widetilde{\alpha})|_s\right) |\sigma\beta - \widetilde{\sigma}\widetilde{\beta}| dQ d\xi dH_s^0 dH_s^1 ds \\
&\le \kappa(1+T) \int_0^t \int_{\mathbb{R}^{8d}} \left(|v-u| + |\widetilde{v}-\widetilde{u}|\right) |\sigma\beta - \widetilde{\sigma}\widetilde{\beta}| dH_s^0 dH_s^1 ds.
\end{aligned}
$$

For $\varepsilon \in (0,\pi)$ let $\Xi_\varepsilon := [\varepsilon,\pi] \times S^{d-2}$ and $\Xi_\varepsilon^c = (0,\varepsilon) \times S^{d-2}$. Setting $\widetilde{\alpha}_0 := \alpha(\widetilde{v},\widetilde{u},\xi_0(v-u,\widetilde{v}-\widetilde{u},\xi))$ observe that, by Lemma 4.3 the function $\xi \longmapsto \xi_0(v-u,\widetilde{v}-\widetilde{u},\xi)$ has Jacobian equal to 1 so that we are allowed to insert it into the integral. This gives

$$
\begin{aligned}
J_1 &= \int_0^t \int_{\Xi\times\mathbb{R}^{8d}} \left(\sigma\beta \wedge \widetilde{\sigma}\widetilde{\beta}\right) S(-s) \Big\{ \psi(r,v+\alpha) - \psi(\widetilde{r},\widetilde{v}+\widetilde{\alpha}_0) \\
&\quad - \psi(r,v) + \psi(\widetilde{r},\widetilde{v})\Big\} dQ d\xi dH_s^0 dH_s^1 ds \\
&\le (1+T) \int_0^t \int_{\mathbb{R}^{8d}\times\Xi_\varepsilon^c} \left(\sigma\beta \wedge \widetilde{\sigma}\widetilde{\beta}\right) \left(|\alpha| + |\widetilde{\alpha}_0|\right) dQ d\xi dH_s^0 dH_s^1 ds \\
&\quad + \int_0^t \int_{\mathbb{R}^{8d}\times\Xi_\varepsilon} \left(\sigma\beta \wedge \widetilde{\sigma}\widetilde{\beta}\right) \Big\{ |(r,v+\alpha) - (\widetilde{r},\widetilde{v}+\widetilde{\alpha}_0)|_s \\
&\quad - |(r,v) - (\widetilde{r},\widetilde{v})|_s \Big\} dQ d\xi dH_s^0 dH_s^1 ds \\
&\quad + \int_0^t \int_{\mathbb{R}^{8d}\times\Xi_\varepsilon} \left(\sigma\beta \wedge \widetilde{\sigma}\widetilde{\beta}\right) \Big\{ |(r,v) - (\widetilde{r},\widetilde{v})|_s \\
&\quad - S(-s)\left(\psi(r,v) - \psi(\widetilde{r},\widetilde{v})\right) \Big\} dQ d\xi dH_s^0 dH_s^1 ds \\
&=: J_1^{(1)} + J_1^{(2)} + J_1^{(3)}.
\end{aligned}
$$

Using $|(0,\alpha)-(0,\widetilde{\alpha})|_s \le (1+T)|\alpha-\widetilde{\alpha}_0|$ and then (4.4) gives

$$\begin{aligned} J_1^{(2)} &\le \int_0^t \int_{\mathbb{R}^{8d}\times\Xi_\varepsilon} \left(\sigma\beta \wedge \widetilde{\sigma}\widetilde{\beta}\right) |(0,\alpha)-(0,\widetilde{\alpha}_0)|_s dQ d\xi dH_s^0 dH_s^1 ds \\ &\le (1+T) \int_0^t \int_{\mathbb{R}^{8d}\times\Xi_\varepsilon} \left(\sigma\beta \wedge \widetilde{\sigma}\widetilde{\beta}\right) |\alpha-\widetilde{\alpha}_0| dQ d\xi dH_s^0 dH_s^1 ds \\ &\le 2\kappa(1+T) \int_0^t \int_{\mathbb{R}^{8d}} \left(\sigma\beta \wedge \widetilde{\sigma}\widetilde{\beta}\right) (|v-\widetilde{v}|+|u-\widetilde{u}|)\, dH_s^0 dH_s^1 ds. \end{aligned}$$

Setting $c_\varepsilon := \int_{\Xi_\varepsilon^c} \theta dQ(\theta) d\xi$, we obtain from the basic estimate $|\alpha(u,v,\theta,\xi)| \le \theta|u-v|$ and $H_s^0, H_s^1 \in \mathcal{H}(\mu_s,\nu_s)$

$$\begin{aligned} J_1^{(1)} &\le (1+T)c_\varepsilon \int_0^t \int_{\mathbb{R}^{8d}} \left(\sigma\beta \wedge \widetilde{\sigma}\widetilde{\beta}\right) (|v-u|+|\widetilde{v}-\widetilde{u}|)\, dH_s^0 dH_s^1 ds \\ &\le (1+T)c_\varepsilon \int_0^t \int_{\mathbb{R}^{8d}} \sigma\beta(|v|+|u|) dH_s^0 dH_s^1 ds \\ &\quad + (1+T)c_\varepsilon \int_0^t \int_{\mathbb{R}^{8d}} \widetilde{\sigma}\widetilde{\beta}(|\widetilde{v}|+|\widetilde{u}|) dH_s^0 dH_s^1 ds \\ &= (1+T)c_\varepsilon \int_0^t \int_{\mathbb{R}^{4d}} \sigma\beta\,(|v|+|u|)\,(d\mu_s(q,u)d\mu_s(r,v)+d\nu_s(q,u)d\nu_s(r,v))\, ds \\ &=: c_\varepsilon h(t). \end{aligned}$$

For $J_1^{(3)}$ we break up the integrand into two parts, namely when $\sigma\beta < N$ and $\sigma\beta \ge N$ where $N \ge 1$. With $\kappa_\varepsilon := Q([\varepsilon,\pi))|S^{d-2}|$ we obtain

$$\begin{aligned} J_1^{(3)} &\le N\kappa_\varepsilon \int_0^t \int_{\mathbb{R}^{8d}} \Big\{ |(r,v)-(\widetilde{r},\widetilde{v})|_s - S(-s)\left(\psi(r,v)-\psi(\widetilde{r},\widetilde{v})\right) \Big\} dH_s^0 dH_s^1 ds \\ &\quad + \kappa_\varepsilon \int_0^t \int_{\mathbb{R}^{8d}} \mathbb{1}_{\{\sigma\beta\ge N\}} \left(\sigma\beta \wedge \widetilde{\sigma}\widetilde{\beta}\right) \Big\{ |(r,v)-(\widetilde{r},\widetilde{v})|_s \\ &\quad - S(-s)\left(\psi(r,v)-\psi(\widetilde{r},\widetilde{v})\right) \Big\} dH_s^0 dH_s^1 ds \\ &\le N\kappa_\varepsilon \int_0^t W_1^s(\mu_s,\nu_s) ds - \kappa_\varepsilon N \int_0^t \langle S(-s)\psi, \mu_s-\nu_s\rangle ds \\ &\quad + 2\kappa_\varepsilon \int_0^t \int_{\mathbb{R}^{8d}} \mathbb{1}_{\{\sigma\beta\ge N\}} \sigma\beta |(r,v)-(\widetilde{r},\widetilde{v})|_s dH_s^0 dH_s^1 ds \end{aligned}$$

where we have used the fact that $\|S(-s)\psi\|_s \le 1$, equation (4.3) and $H_s^0, H_s^1 \in \mathcal{H}(\mu_s,\nu_s)$. For the last term we apply $|(r,v)-(\widetilde{r},\widetilde{v})|_s \le (1+T)\left(|r|+|v|+|\widetilde{r}|+|\widetilde{v}|\right)$ and then $H_s^0, H_s^1 \in \mathcal{H}(\mu_s,\nu_s)$ to obtain

$$J_1^{(3)} \le \kappa_\varepsilon N \int_0^t W_1^s(\mu_s,\nu_s) ds - \kappa_\varepsilon N \int_0^t \langle S(-s)\psi, \mu_s-\nu_s\rangle ds + \kappa_\varepsilon g_N(t)$$

where

$$g_N(t) = 2(1+T) \int_0^t \int_{\mathbb{R}^{4d}} \sigma\beta \mathbb{1}_{\{\sigma\beta\ge N\}} \left(|r|+|v|\right) (d\mu_s(q,u)d\mu_s(r,v)+d\nu_s(q,u)d\nu_s(r,v))\, ds.$$

Collecting all inequalities gives

$$\begin{aligned}\langle S(-t)\psi, \mu_t - \nu_t\rangle &\le W_1(\mu_0,\nu_0) + 2\kappa(1+T)\int_0^t \int_{\mathbb{R}^{8d}} \Psi dH_s^0 dH_s^1 ds \\ &\quad + \int_0^t (\kappa_\varepsilon g_N(s) + c_\varepsilon h(s))ds + N\kappa_\varepsilon \int_0^t W_1^s(\mu_s,\nu_s)ds \\ &\quad - \kappa_\varepsilon N \int_0^t \langle S(-s)\psi, \mu_s - \nu_s\rangle ds.\end{aligned}$$

This yields

$$\begin{aligned}\langle S(-t)\psi, \mu_t - \nu_t\rangle e^{\kappa_\varepsilon N t} &\le W_1(\mu_0,\nu_0) + 2\kappa(1+T)\int_0^t e^{\kappa_\varepsilon N s} \int_{\mathbb{R}^{8d}} \Psi dH_s^0 dH_s^1 ds \\ &\quad + \int_0^t e^{\kappa_\varepsilon N s}(\kappa_\varepsilon g_N(s) + c_\varepsilon h(s))ds + N\kappa_\varepsilon \int_0^t e^{\kappa_\varepsilon N s} W_1^s(\mu_s,\nu_s)ds.\end{aligned}$$

Taking the supremum over $\psi \in \mathrm{Lip}(\mathbb{R}^{2d})$ with $\|\psi\|_0 \le 1$, using (4.1) and, then applying the generalized Gronwall inequality gives

$$\begin{aligned}W_1^t(\mu_t,\nu_t) &\le W_1(\mu_0,\nu_0) + 2\kappa(1+T)\int_0^t \int_{\mathbb{R}^{8d}} \Psi dH_s^0 dH_s^1 ds \\ &\quad + \int_0^t \left(\kappa_\varepsilon g_N(s) + c_\varepsilon h(s)\right) ds.\end{aligned}$$

Taking first $N \to \infty$ and then $\varepsilon \to 0$ proves the assertion. □

5. Proof of Theorem 2.4

Assume that $\gamma \in [0,2]$ and let δ be given by (2.12). Let us first show that under the conditions of Theorem 2.4, also (4.2) is satisfied. Indeed, using $\sigma(|v-u|) \le C\langle v\rangle^2\langle u\rangle^2$ this follows from

$$\begin{aligned}(|v|+|u|+|r|+|q|)\sigma(|v-u|) &\le C(|v|+|u|)\langle v\rangle^2\langle u\rangle^2 + C(|r|+|q|)\langle v\rangle^2\langle u\rangle^2 \\ &\le C\langle v\rangle^3\langle u\rangle^3 + C\langle r\rangle\langle q\rangle\langle v\rangle\langle u\rangle \\ &\le C\langle v\rangle^3\langle u\rangle^3 + C\left(\langle r\rangle^{1+\delta} + \langle v\rangle^{2+\frac{2}{\delta}}\right)\left(\langle q\rangle^{1+\delta} + \langle u\rangle^{2+\frac{2}{\delta}}\right),\end{aligned}$$

where we have used Jensen's inequality

$$\langle r\rangle\langle v\rangle^2 \le \frac{1}{1+\delta}\langle r\rangle^{1+\delta} + \frac{\delta}{1+\delta}\langle v\rangle^{2\frac{1+\delta}{\delta}}.$$

Thus we can apply the coupling inequality. In order to estimate Ψ as defined in Proposition 4.2, we need the following lemma.

Lemma 5.1. *There exists a constant $C = C(\gamma,\sigma) > 0$ such that the following holds:*

(a) *For all $v, u, \widetilde{v}, \widetilde{u} \in \mathbb{R}^d$*

$$\begin{aligned}&(|v-u| + |\widetilde{v}-\widetilde{u}|)\,|\sigma(|v-u|) - \sigma(|\widetilde{v}-\widetilde{u}|)| \\ &\qquad \le C\left(|v-u|^\gamma + |\widetilde{v}-\widetilde{u}|^\gamma\right)\left(|v-\widetilde{v}| + |u-\widetilde{u}|\right).\end{aligned}$$

Note that for $\gamma = 0$ the left-hand side is zero and hence the assertion is trivially satisfied.

(b) *For all $v, u, \widetilde{v}, \widetilde{u}$*

$$\begin{aligned}&(|v-u| + |\widetilde{v}-\widetilde{u}|)\,|\sigma(|v-u|)\beta(r-q) - \sigma(|\widetilde{v}-\widetilde{u}|)\beta(\widetilde{r}-\widetilde{q})| \\ &\quad \le C\left(\langle v\rangle^{\gamma} + \langle u\rangle^{\gamma} + \langle \widetilde{v}\rangle^{\gamma} + \langle \widetilde{u}\rangle^{\gamma}\right)(|v-\widetilde{v}| + |u-\widetilde{u}|) \\ &\qquad + C\left(\langle v\rangle^{1+\gamma} + \langle u\rangle^{1+\gamma} + \langle \widetilde{v}\rangle^{1+\gamma} + \langle \widetilde{u}\rangle^{1+\gamma}\right)(|r-\widetilde{r}| + |q-\widetilde{q}|)\,.\end{aligned}$$

Proof. (a) We use, for $x, y \ge 0$ and $a, b > 0$, the elementary inequality

$$c_{a,b}|x^{a+b} - y^{a+b}| \le (x^a + y^a)|x^b - y^b| \le C_{a,b}|x^{a+b} - y^{a+b}|$$

with some constants $c_{a,b}, C_{a,b} > 0$ (see e.g. [14] for a similar application) to obtain

$$\begin{aligned}&(|v-u| + |\widetilde{v}-\widetilde{u}|)\,|\sigma(|v-u|) - \sigma(|\widetilde{v}-\widetilde{u}|)| \\ &\quad \le C\,(|v-u| + |\widetilde{v}-\widetilde{u}|)\,\big||v-u|^{\gamma} - |\widetilde{v}-\widetilde{u}|^{\gamma}\big| \\ &\quad \le C\left||v-u|^{1+\gamma} - |\widetilde{v}-\widetilde{u}|^{1+\gamma}\right| \le C\left(|v-u|^{\gamma} + |\widetilde{v}-\widetilde{u}|^{\gamma}\right)(|v-\widetilde{v}| + |u-\widetilde{u}|)\,.\end{aligned}$$

(b) We estimate by (a)

$$\begin{aligned}&|v-u||\sigma(|v-u|)\beta(r-q) - \sigma(|\widetilde{v}-\widetilde{u}|)\beta(\widetilde{r}-\widetilde{q})| \\ &\quad \le |v-u|\left(\beta(\widetilde{r}-\widetilde{q})|\sigma(|v-u|) - \sigma(|\widetilde{v}-\widetilde{u}|)| + \sigma(|v-u|)|\beta(r-q) - \beta(\widetilde{r}-\widetilde{q})|\right) \\ &\quad \le C\left(\langle v\rangle^{\gamma} + \langle u\rangle^{\gamma} + \langle \widetilde{v}\rangle^{\gamma} + \langle \widetilde{u}\rangle^{\gamma}\right)(|v-\widetilde{v}| + |u-\widetilde{u}|) \\ &\qquad + C\left(\langle v\rangle^{1+\gamma} + \langle u\rangle^{1+\gamma}\right)(|r-\widetilde{r}| + |q-\widetilde{q}|)\,.\end{aligned}$$

The second term can be estimated in the same way. ◻

From this lemma we deduce the following estimate on the function Ψ.

Lemma 5.2. *There exists a constant $C = C(\delta, \gamma, \sigma) > 0$ such that*

$$\begin{aligned}\Psi &\le C\left(e^{\delta\langle u\rangle^{1+\gamma}} + e^{\delta\langle \widetilde{u}\rangle^{1+\gamma}}\right)(|v-\widetilde{v}| + |r-\widetilde{r}|) \\ &\quad + C\left(e^{\delta\langle v\rangle^{1+\gamma}} + e^{\delta\langle \widetilde{v}\rangle^{1+\gamma}}\right)(|u-\widetilde{u}| + |q-\widetilde{q}|) \\ &\quad + C\left(\langle v\rangle^{1+\gamma} + \langle \widetilde{v}\rangle^{1+\gamma}\right)(|v-\widetilde{v}| + |r-\widetilde{r}|) \\ &\quad + C\left(\langle u\rangle^{1+\gamma} + \langle \widetilde{u}\rangle^{1+\gamma}\right)(|u-\widetilde{u}| + |q-\widetilde{q}|)\,.\end{aligned}$$

Proof. Using the inequality

$$\begin{aligned}&\sigma(|v-u|) \wedge \sigma(|\widetilde{v}-\widetilde{u}|)\,(|v-\widetilde{v}| + |u-\widetilde{u}|) \\ &\quad \le C\left(\langle u\rangle^{\gamma} + \langle v\rangle^{\gamma} + \langle \widetilde{u}\rangle^{\gamma} + \langle \widetilde{v}\rangle^{\gamma}\right)(|v-\widetilde{v}| + |u-\widetilde{u}|)\end{aligned}$$

we obtain from Lemma 5.1

$$\begin{aligned}\Psi &\le C\left(\langle u\rangle^{\gamma} + \langle v\rangle^{\gamma} + \langle \widetilde{u}\rangle^{\gamma} + \langle \widetilde{v}\rangle^{\gamma}\right)(|v-\widetilde{v}| + |u-\widetilde{u}|) \\ &\quad + C\left(\langle u\rangle^{1+\gamma} + \langle v\rangle^{1+\gamma} + \langle \widetilde{u}\rangle^{1+\gamma} + \langle \widetilde{v}\rangle^{1+\gamma}\right)(|r-\widetilde{r}| + |q-\widetilde{q}|) \\ &\le C\left(\langle u\rangle^{1+\gamma} + \langle \widetilde{u}\rangle^{1+\gamma}\right)(|v-\widetilde{v}| + |r-\widetilde{r}|) \\ &\quad + C\left(\langle v\rangle^{1+\gamma} + \langle \widetilde{v}\rangle^{1+\gamma}\right)(|u-\widetilde{u}| + |q-\widetilde{q}|) \\ &\quad + C\left(\langle v\rangle^{1+\gamma} + \langle \widetilde{v}\rangle^{1+\gamma}\right)(|v-\widetilde{v}| + |r-\widetilde{r}|)\end{aligned}$$

$$+ C\left(\langle u\rangle^{1+\gamma} + \langle\widetilde{u}\rangle^{1+\gamma}\right)\left(|u-\widetilde{u}| + |q-\widetilde{q}|\right),$$

where $C = C(\gamma,\sigma) > 0$ is some constant. Estimating the polynomials by the exponential function yields the assertion. □

The coupling inequality combined with Lemma 5.2 gives for some constant $K > 0$

$$W_1^t(\mu_t,\nu_t) \le W_1(\mu_0,\nu_0) + K\mathcal{C}_\gamma(T,\mu+\nu,\delta)\int_0^t W_1^s(\mu_s,\nu_s)ds$$
$$+ K\mathcal{C}_\gamma(T,\mu+\nu,\delta)\int_0^t\int_{\mathbb{R}^{2d}}\left(\langle v\rangle^{1+\gamma} + \langle\widetilde{v}\rangle^{1+\gamma}\right)\left(|v-\widetilde{v}| + |r-\widetilde{r}|\right)dH_s ds,$$

where we have used $|v-\widetilde{v}| + |r-\widetilde{r}| \le (1+T)|(r,v)-(\widetilde{r},\widetilde{v})|_s$, (4.3) and $H_s = H_s(dr,dv;d\widetilde{r},d\widetilde{v}) \in \mathcal{H}(\mu_s,\nu_s)$. The proof of Theorem 2.4 is complete once we have shown the following Lemma.

Lemma 5.3. *There exists a constant $K > 0$ such that for each $s \in [0,t]$ we have*

$$\int_{\mathbb{R}^{4d}}\left(\langle v\rangle^{1+\gamma} + \langle\widetilde{v}\rangle^{1+\gamma}\right)|r-\widetilde{r}|dH_s(r,v;\widetilde{r},\widetilde{v})$$
$$\le K\mathcal{C}_\gamma(T,\mu+\nu,\delta)\left(1 + |\ln(W_1^s(\mu_s,\nu_s))|\right)W_1^s(\mu_s,\nu_s) \tag{5.1}$$

and

$$\int_{\mathbb{R}^{4d}}\left(\langle v\rangle^{1+\gamma} + \langle\widetilde{v}\rangle^{1+\gamma}\right)|v-\widetilde{v}|dH_s(r,v;\widetilde{r},\widetilde{v})$$
$$\le K\mathcal{C}_\gamma(T,\mu+\nu,\delta)\left(1 + |\ln(W_1^s(\mu_s,\nu_s))|\right)W_1^s(\mu_s,\nu_s). \tag{5.2}$$

A similar estimate to (5.2) was shown in [15, p.820], while (5.1) did not appear in the space-homogeneous setting studied there.

Proof. Here and below we denote by $K > 0$ some generic constant (independent of $(\mu_t)_{t\in[0,T]}$ and $(\nu_t)_{t\in[0,T]}$) which may vary from line to line. For $b > 0$, we obtain

$$\int_{\mathbb{R}^{4d}}\left(\langle v\rangle^{1+\gamma} + \langle\widetilde{v}\rangle^{1+\gamma}\right)|r-\widetilde{r}|dH_s(r,v;\widetilde{r},\widetilde{v})$$
$$= \int_{\mathbb{R}^{4d}}\mathbb{1}_{\{\langle v\rangle\le b,\ \langle\widetilde{v}\rangle\le b\}}\left(\langle v\rangle^{1+\gamma} + \langle\widetilde{v}\rangle^{1+\gamma}\right)|r-\widetilde{r}|dH_s(r,v;\widetilde{r},\widetilde{v})$$
$$+ \int_{\mathbb{R}^{4d}}\left(1 - \mathbb{1}_{\{\langle v\rangle\le b,\ \langle\widetilde{v}\rangle\le b\}}\right)\left(\langle v\rangle^{1+\gamma} + \langle\widetilde{v}\rangle^{1+\gamma}\right)|r-\widetilde{r}|dH_s(r,v;\widetilde{r},\widetilde{v})$$
$$\le 2b^{1+\gamma}(1+T)W_1^s(\mu_s,\nu_s) + 2I(b)$$

where we have used $|r-\widetilde{r}| \le (1+T)|(r,v)-(\widetilde{r},\widetilde{v})|_s$ and

$$I(b) := \int_{\mathbb{R}^{4d}}\left(\langle v\rangle^{1+\gamma} + \langle\widetilde{v}\rangle^{1+\gamma}\right)\left(|r| + |\widetilde{r}|\right)\left(\mathbb{1}_{\{\langle v\rangle > b\}} + \mathbb{1}_{\{\langle\widetilde{v}\rangle > b\}}\right)dH_s^1.$$

For K large enough satisfying for all $v,\widetilde{v} \in \mathbb{R}^d$

$$\left(\langle v\rangle^{1+\gamma} + \langle\widetilde{v}\rangle^{1+\gamma}\right)\left(e^{\frac{\delta}{1+\delta}\frac{\delta}{2}\langle v\rangle^{1+\gamma}} + e^{\frac{\delta}{1+\delta}\frac{\delta}{2}\langle\widetilde{v}\rangle^{1+\gamma}}\right) \le K\left(e^{\frac{\delta}{1+\delta}\delta\langle v\rangle^{1+\gamma}} + e^{\frac{\delta}{1+\delta}\delta\langle\widetilde{v}\rangle^{1+\gamma}}\right)$$

we obtain

$$\left(\langle v\rangle^{1+\gamma}+\langle\widetilde{v}\rangle^{1+\gamma}\right)\mathbb{1}_{\{\langle v\rangle>b\}}\leq Ke^{-\frac{\delta}{1+\delta}\frac{\delta}{2}b^{1+\gamma}}\left(e^{\frac{\delta}{1+\delta}\delta\langle v\rangle^{1+\gamma}}+e^{\frac{\delta}{1+\delta}\delta\langle\widetilde{v}\rangle^{1+\gamma}}\right).$$

Estimating the second term in $I(b)$ in the same way gives

$$\begin{aligned}I(b)&\leq Ke^{-\frac{\delta}{1+\delta}\frac{\delta}{2}b^{1+\gamma}}\int_{\mathbb{R}^{4d}}\left(e^{\frac{\delta}{1+\delta}\delta\langle v\rangle^{1+\gamma}}+e^{\frac{\delta}{1+\delta}\delta\langle\widetilde{v}\rangle^{1+\gamma}}\right)(|r|+|\widetilde{r}|)dH_s(r,v;\widetilde{r},\widetilde{v})\\&\leq Ke^{-\frac{\delta}{1+\delta}\frac{\delta}{2}b^{1+\gamma}}\mathcal{C}_\gamma(T,\mu+\nu,\delta),\end{aligned}$$

where we have used $H_s\in\mathcal{H}(\mu_s,\nu_s)$ and

$$\begin{aligned}&\left(e^{\frac{\delta}{1+\delta}\delta\langle v\rangle^{1+\gamma}}+e^{\frac{\delta}{1+\delta}\delta\langle\widetilde{v}\rangle^{1+\gamma}}\right)(|r|+|\widetilde{r}|)\\&\quad\leq\frac{\delta}{1+\delta}\left(e^{\frac{\delta}{1+\delta}\delta\langle v\rangle^{1+\gamma}}+e^{\frac{\delta}{1+\delta}\delta\langle\widetilde{v}\rangle^{1+\gamma}}\right)^{\frac{1+\delta}{\delta}}+\frac{1}{1+\delta}(|r|+|\widetilde{r}|)^{1+\delta}\\&\quad\leq K\left(e^{\frac{\delta}{2}\langle v\rangle^{1+\gamma}}+e^{\frac{\delta}{2}\langle\widetilde{v}\rangle^{1+\gamma}}\right)+K\left(|r|^{1+\delta}+|\widetilde{r}|^{1+\delta}\right).\end{aligned}$$

This gives the estimate

$$\begin{aligned}&\int_{\mathbb{R}^{4d}}\left(\langle v\rangle^{1+\gamma}+\langle\widetilde{v}\rangle^{1+\gamma}\right)|r-\widetilde{r}|dH_s(r,v;\widetilde{r},\widetilde{v})\\&\quad\leq 2b^{1+\gamma}(1+T)W_1^s(\mu_s,\nu_s)+Ke^{-\frac{\delta}{1+\delta}\frac{\delta}{2}b^{1+\gamma}}\mathcal{C}_\gamma(T,\mu+\nu,\delta).\end{aligned}$$

Letting $b^{1+\gamma}=\frac{1+\delta}{\delta}\frac{2}{\delta}\left|\ln\left(W_1^s(\mu_s,\nu_s)\right)\right|$ yields (5.1). The inequality (5.2) can be shown in the same way. ☐

6. Proof of Theorem 2.5

In this section we assume that $\gamma\in(-d,0]$. As before, we first check that (4.2) is satisfied. Indeed we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}(|v|+|u|+|r|+|q|)\sigma(|v-u|)\mu_t(dq,du)\mu_t(dr,dv)\\&\quad\leq K\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}(|v|+|u|+|r|+|q|)|v-u|^\gamma\mu_t(dq,du)\mu_t(dr,dv)\\&\quad\leq K\Lambda(\mu_t)\int_{\mathbb{R}^{2d}}(|v|+|r|)\mu_t(dr,dv)\end{aligned}$$

for some generic constant $K>0$. Since an analogous inequality also holds for ν_t, (4.2) is satisfied and we may apply the coupling inequality. Again, to estimate the expression Ψ appearing in Proposition 4.2 we use the following lemma.

Lemma 6.1. *There exists a constant $C=C(\gamma,\sigma)>0$ such that the following holds:*

(a) *For all $v,u,\widetilde{v},\widetilde{u}\in\mathbb{R}^d$*

$$\begin{aligned}&(|v-u|+|\widetilde{v}-\widetilde{u}|)\,|\sigma(|v-u|)-\sigma(|\widetilde{v}-\widetilde{u}|)|\\&\qquad\leq C\left(|v-u|^\gamma+|\widetilde{v}-\widetilde{u}|^\gamma\right)(|v-\widetilde{v}|+|u-\widetilde{u}|).\qquad(6.1)\end{aligned}$$

(b) *If* $\gamma \in (-1, 0]$, *then for all* $v, u, \widetilde{v}, \widetilde{u} \in \mathbb{R}^d$

$$\begin{aligned}
&(|v-u| + |\widetilde{v}-\widetilde{u}|)\, |\sigma(|v-u|)\beta(r-q) - \sigma(|\widetilde{v}-\widetilde{u}|)\beta(\widetilde{r}-\widetilde{q})| \\
&\quad \le C\left(\langle v\rangle^{1+\gamma} + \langle u\rangle^{1+\gamma} + \langle \widetilde{v}\rangle^{1+\gamma} + \langle \widetilde{u}\rangle^{1+\gamma}\right)(|r-\widetilde{r}| + |q-\widetilde{q}|) \\
&\qquad + C\left(|v-u|^{\gamma} + |\widetilde{v}-\widetilde{u}|^{\gamma}\right)(|v-\widetilde{v}| + |u-\widetilde{u}|).
\end{aligned}$$

(c) *If* $\gamma \in (-d, -1]$, *then for all* $v, u, \widetilde{v}, \widetilde{u} \in \mathbb{R}^d$

$$\begin{aligned}
&(|v-u| + |\widetilde{v}-\widetilde{u}|)\, |\sigma(|v-u|)\beta(r-q) - \sigma(|\widetilde{v}-\widetilde{u}|)\beta(\widetilde{r}-\widetilde{q})| \\
&\quad \le C\left(|v-u|^{1+\gamma} + |\widetilde{v}-\widetilde{u}|^{1+\gamma}\right)(|r-\widetilde{r}| + |q-\widetilde{q}|) \\
&\qquad + C\left(|v-u|^{\gamma} + |\widetilde{v}-\widetilde{u}|^{\gamma}\right)(|v-\widetilde{v}| + |u-\widetilde{u}|).
\end{aligned}$$

Proof. (a) Following [15, p. 821] we obtain

$$\begin{aligned}
&|v-u|\, |\sigma(|v-u|) - \sigma(|\widetilde{v}-\widetilde{u}|)| \\
&\quad \le c_\sigma \left(|v-u| \wedge |\widetilde{v}-\widetilde{u}| + ||v-u| - |\widetilde{v}-\widetilde{u}||\right) ||v-u|^{\gamma} - |\widetilde{v}-\widetilde{u}|^{\gamma}| \\
&\quad \le C|\gamma| \left(|v-u| \wedge |\widetilde{v}-\widetilde{u}|\right)^{\gamma-1} ||v-u|^{\gamma} - |\widetilde{v}-\widetilde{u}|^{\gamma}|\, |v-u| \wedge |\widetilde{v}-\widetilde{u}| \\
&\qquad + C\left(|v-u|^{\gamma} \vee |\widetilde{v}-\widetilde{u}|^{\gamma}\right) ||v-u|^{\gamma} - |\widetilde{v}-\widetilde{u}|^{\gamma}| \\
&\quad \le C(1+|\gamma|)\left(|v-u|^{\gamma} + |\widetilde{v}-\widetilde{u}|^{\gamma}\right)(|v-\widetilde{v}| + |u-\widetilde{u}|).
\end{aligned}$$

The other term can be estimated in the same way.

(b) By (a) it follows that

$$\begin{aligned}
&|v-u||\sigma(|v-u|)\beta(r-q) - \sigma(|\widetilde{v}-\widetilde{u}|)\beta(\widetilde{r}-\widetilde{q})| \\
&\quad \le |v-u|\left(\sigma(|v-u|)|\beta(r-q) - \beta(\widetilde{r}-\widetilde{q})| + \beta(\widetilde{r}-\widetilde{q})|\sigma(|v-u|) - \sigma(|\widetilde{v}-\widetilde{u}|)|\right) \\
&\quad \le C\left(\langle v\rangle^{1+\gamma} + \langle u\rangle^{1+\gamma}\right)(|r-\widetilde{r}| + |q-\widetilde{q}|) + C\left(|v-u|^{\gamma} + |\widetilde{v}-\widetilde{u}|^{\gamma}\right)(|v-\widetilde{v}| + |u-\widetilde{u}|).
\end{aligned}$$

The other term is estimated in the same way.

(c) In this case we obtain

$$\begin{aligned}
&|v-u||\sigma(|v-u|)\beta(r-q) - \sigma(|\widetilde{v}-\widetilde{u}|)\beta(\widetilde{r}-\widetilde{q})| \\
&\quad \le |v-u|\left(\sigma(|v-u|)|\beta(r-q) - \beta(\widetilde{r}-\widetilde{q})| + \beta(\widetilde{r}-\widetilde{q})|\sigma(|v-u|) - \sigma(|\widetilde{v}-\widetilde{u}|)|\right) \\
&\quad \le C|v-u|^{1+\gamma}(|r-\widetilde{r}| + |q-\widetilde{q}|) + C\left(|v-u|^{\gamma} + |\widetilde{v}-\widetilde{u}|^{\gamma}\right)(|v-\widetilde{v}| + |u-\widetilde{u}|)
\end{aligned}$$

and similarly for the second term. □

From this we deduce the following estimates for Ψ.

Lemma 6.2. *There exists a constant* $C > 0$ *such that*

(a) *If* $\gamma \in (-d, -1]$, *then*

$$\begin{aligned}
\Psi &\le C\left(|v-u|^{1+\gamma} + |\widetilde{v}-\widetilde{u}|^{1+\gamma}\right)(|r-\widetilde{r}| + |q-\widetilde{q}|) \\
&\quad + C\left(|v-u|^{\gamma} + |\widetilde{v}-\widetilde{u}|^{\gamma}\right)(|v-\widetilde{v}| + |u-\widetilde{u}|).
\end{aligned}$$

(b) *If* $\gamma \in (-1, 0)$, *then*

$$\begin{aligned}
\Psi &\le C\left(\langle v\rangle^{1+\gamma} + \langle u\rangle^{1+\gamma} + \langle \widetilde{v}\rangle^{1+\gamma} + \langle \widetilde{u}\rangle^{1+\gamma}\right)(|r-\widetilde{r}| + |q-\widetilde{q}|) \\
&\quad + C\left(|v-u|^{\gamma} + |\widetilde{v}-\widetilde{u}|^{\gamma}\right)(|v-\widetilde{v}| + |u-\widetilde{u}|)
\end{aligned}$$

Proof. Lemma 6.2 is now a consequence of

$$\begin{aligned}\sigma(|v-u|) \wedge \sigma(|\widetilde{v}-\widetilde{u}|)\,(|v-\widetilde{v}|+|u-\widetilde{u}|) \\ \leq C\,(|v-u|^{\gamma}+|\widetilde{v}-\widetilde{u}|^{\gamma})\,(|v-\widetilde{v}|+|u-\widetilde{u}|)\end{aligned}$$

and Lemma 6.1 from the "Appendix". □

Below we treat the cases $\gamma \in (-d,-1]$ and $\gamma \in (-1,0)$ seperately.

6.1. Case $\gamma \in (-d,-1]$

We obtain from the general coupling inequality together with Lemma 6.2 and $H_s \in \mathcal{H}(\mu_s,\nu_s)$

$$\begin{aligned}
&W_1^t(\mu_t,\nu_t) \leq W_1(\mu_0,\nu_0)\\
&\quad + K\int_0^t \int_{\mathbb{R}^{8d}} (|v-u|^{\gamma}+|\widetilde{v}-\widetilde{u}|^{\gamma})\,(|v-\widetilde{v}|+|u-\widetilde{u}|)\,dH_s(q,u;\widetilde{q},\widetilde{u})dH_s(r,v;\widetilde{r},\widetilde{v})ds\\
&\quad + K\int_0^t \int_{\mathbb{R}^{8d}} \left(|v-u|^{1+\gamma}+|\widetilde{v}-\widetilde{u}|^{1+\gamma}\right)(|r-\widetilde{r}|+|q-\widetilde{q}|)\,dH_s(q,u;\widetilde{q},\widetilde{u})dH_s(r,v;\widetilde{r},\widetilde{v})ds\\
&\leq W_1(\mu_0,\nu_0) + K\int_0^t \int_{\mathbb{R}^{4d}} \Lambda(s,\mu,\nu)|v-\widetilde{v}|dH_s(r,v;\widetilde{r},\widetilde{v})\\
&\quad + K\int_0^t \int_{\mathbb{R}^{4d}} \widetilde{\Lambda}(s,\mu,\nu)|r-\widetilde{r}|dH_s(r,v;\widetilde{r},\widetilde{v})\\
&\leq W_1(\mu_0,\nu_0) + K\int_0^t \left(\Lambda(s,\mu,\nu)+\widetilde{\Lambda}(s,\mu,\nu)\right)W_1^s(\mu_s,\nu_s)ds,
\end{aligned}$$

where we have used $|v-\widetilde{v}| \leq |(r,v)-(\widetilde{r},\widetilde{v})|_s$ and $|r-\widetilde{r}| \leq (1+T)|(r,v)-(\widetilde{r},\widetilde{v})|_s$ and

$$\widetilde{\Lambda}(s,\mu,\nu) = \sup_{u\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} |v-u|^{1+\gamma}d(\mu_s+\nu_s)(r,v).$$

Since $1+\gamma \leq 0$ we obtain $|v-u|^{1+\gamma} \leq 1+|v-u|^{\gamma}$ and hence $\widetilde{\Lambda}(s,\mu,\nu) \leq 1+\Lambda(s,\mu,\nu)$ gives

$$W_1^t(\mu_t,\nu_t) \leq W_1(\mu_0,\nu_0) + K\int_0^t (1+\Lambda(s,\mu,\nu))\,W_1^s(\mu_s,\nu_s)ds.$$

The assertion follows from the classical Gronwall lemma.

6.2. Case $\gamma \in (-1,0)$

Proceeding as before we obtain from the coupling inequality and Lemma 6.2

$$\begin{aligned}
&W_1^t(\mu_t,\nu_t) \leq W_1(\mu_0,\nu_0)\\
&\quad + K\int_0^t \int_{\mathbb{R}^{8d}} (|v-u|^{\gamma}+|\widetilde{v}-\widetilde{u}|^{\gamma})\,(|v-\widetilde{v}|+|u-\widetilde{u}|)\,dH_s(q,u;\widetilde{q},\widetilde{u})dH_s(r,v;\widetilde{r},\widetilde{v})ds\\
&\quad + K\int_0^t \int_{\mathbb{R}^{8d}} \left(\langle u\rangle^{1+\gamma}+\langle\widetilde{u}\rangle^{1+\gamma}\right)|r-\widetilde{r}|dH_s(q,u;\widetilde{q},\widetilde{u})dH_s(r,v;\widetilde{r},\widetilde{v})ds\\
&\quad + K\int_0^t \int_{\mathbb{R}^{8d}} \left(\langle v\rangle^{1+\gamma}+\langle\widetilde{v}\rangle^{1+\gamma}\right)|r-\widetilde{r}|dH_s(r,v;\widetilde{r},\widetilde{v})ds\\
&\leq W_1(\mu_0,\nu_0) + K\int_0^t (1+\Lambda(s,\mu,\nu))\,W_1^s(\mu_s,\nu_s)ds
\end{aligned}$$

$$+ K \int_0^t \int_{\mathbb{R}^{4d}} \left(\langle v \rangle^{1+\gamma} + \langle \widetilde{v} \rangle^{1+\gamma} \right) |r - \widetilde{r}| dH_s(r, v; \widetilde{r}, \widetilde{v}) ds$$
$$\leq W_1(\mu_0, \nu_0) + K\mathcal{C}_\gamma(T, \mu + \nu, \delta) \int_0^t W_1^s(\mu_s, \nu_s)(1 + \Lambda(s, \mu, \nu) + |\ln(W_1^s(\mu_s, \nu_s))|) ds,$$

where in the last inequality we have used similar arguments as in the proof of (5.1). This implies the assertion.

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Appendix A: Nonlinear Gronwall inequality

The following lemma is due to [12, Lemma 5.2.1, p. 89].

Lemma A.1. *Let ρ be a nonnegative bounded function on $[0, T]$, $a \in [0, \infty)$ and g be a strictly positive and non-decreasing function on $(0, \infty)$. Suppose that $\int_0^1 \frac{dx}{g(x)} = \infty$ and*

$$\rho(t) \leq a + \int_0^t g(\rho(s)) ds, \quad t \in [0, T].$$

Then

(a) *If $a = 0$, then $\rho(t) = 0$ for all $t \in [0, T]$.*
(b) *If $a > 0$, then $G(a) - G(\rho(t)) \leq t$ where $G(x) = \int_x^1 \frac{dy}{g(y)}$.*

References

[1] Albeverio, S., Rüdiger, B., Sundar, P.: The Enskog process. J. Stat. Phys. **167**(1), 90–122 (2017)

[2] Alexandre, R.: A review of Boltzmann equation with singular kernels. Kinet. Relat. Models **2**(4), 551–646 (2009)

[3] Alexandre, R., Morimoto, Y., Ukai, S., Chao-Jiang, X., Yang, T.: The Boltzmann equation without angular cutoff in the whole space: II, Global existence for hard potential. Anal. Appl. (Singap.) **9**(2), 113–134 (2011)

[4] Alexandre, R., Morimoto, Y., Ukai, S., Chao-Jiang, X., Yang, T.: The Boltzmann equation without angular cutoff in the whole space: qualitative properties of solutions. Arch. Ration. Mech. Anal. **202**(2), 599–661 (2011)

[5] Alexandre, R., Morimoto, Y., Ukai, S., Chao-Jiang, X., Yang, T.: Global existence and full regularity of the Boltzmann equation without angular cutoff. Commun. Math. Phys. **304**(2), 513–581 (2011)

[6] Alexandre, R., Morimoto, Y., Ukai, S., Chao-Jiang, X., Yang, T.: The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential. J. Funct. Anal. **262**(3), 915–1010 (2012)

[7] Arkeryd, L.: On the Enskog equation with large initial data. SIAM J. Math. Anal. **21**(3), 631–646 (1990)

[8] Arkeryd, L., Cercignani, C.: Global existence in L^1 for the Enskog equation and convergence of the solutions to solutions of the Boltzmann equation. J. Stat. Phys. **59**(3–4), 845–867 (1990)

[9] Cercignani, C.: The Grad limit for a system of soft spheres. Commun. Pure Appl. Math. **36**(4), 479–494 (1983)

[10] Cercignani, C.: The Boltzmann Equation and Its Applications, Applied Mathematical Sciences, vol. 67. Springer, New York (1988)

[11] Cercignani, C., Illner, R., Pulvirenti, M.: The Mathematical Theory of Dilute Gases, Applied Mathematical Sciences, Teil 106. Springer, New York (1994)

[12] Chemin, J.-Y.: Fluides parfaits incompressibles. Astérisque **177**(230), MR 1340046 (1995)

[13] Desvillettes, L., Mouhot, C.: Stability and uniqueness for the spatially homogeneous Boltzmann equation with long-range interactions. Arch. Ration. Mech. Anal. **193**(2), 227–253 (2009)

[14] Fournier, N.: Finiteness of entropy for the homogeneous Boltzmann equation with measure initial condition. Ann. Appl. Probab. **25**(2), 860–897 (2015)

[15] Fournier, N., Mouhot, C.: On the well-posedness of the spatially homogeneous Boltzmann equation with a moderate angular singularity. Commun. Math. Phys. **289**(3), 803–824 (2009)

[16] Friesen, M., Rüdiger, B., Sundar, P.: The Enskog process for hard and soft potentials. NoDEA Nonlinear Differ. Equ. Appl. **26**(3), 42 (2019)

[17] Horowitz, J., Karandikar, R.L.: Martingale problems associated with the Boltzmann equation, Seminar on Stochastic Processes, 1989 (San Diego, CA, 1989), Progr. Probab., vol. 18, Birkhäuser Boston, Boston, MA, pp. 75–122 (1990)

[18] Lu, X., Mouhot, C.: On measure solutions of the Boltzmann equation, part I: moment production and stability estimates. J. Differ. Equ. **252**(4), 3305–3363 (2012)

[19] Lu, X., Mouhot, C.: On measure solutions of the Boltzmann equation, Part II: rate of convergence to equilibrium. J. Differ. Equ. **258**(11), 3742–3810 (2015)

[20] Povzner, A.Y.: On the Boltzmann equation in the kinetic theory of gases. Mat. Sb. (N.S.) **58**(100), 65–86 (1962)

[21] Rezakhanlou, F.: A stochastic model associated with Enskog equation and its kinetic limit. Commun. Math. Phys. **232**(2), 327–375 (2003)

[22] Tanaka, H.: Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. Z. Wahrsch. Verw. Gebiete **46**(1), 67–105 (1978/79)

[23] Toscani, G., Bellomo, N.: The Enskog-Boltzmann equation in the whole space R^3: some global existence, uniqueness and stability results. Comput. Math. Appl. **13**(9–11), 851–859 (1987)

[24] Villani, C.: A review of mathematical topics in collisional kinetic theory, Handbook of mathematical fluid dynamics, Vol. I, North-Holland, Amsterdam, pp. 71–305 (2002)

[25] Villani, C.: Optimal transport, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338. Springer, Berlin (2009)

[26] Xu, L.: Uniqueness and propagation of chaos for the Boltzmann equation with moderately soft potentials. Ann. Appl. Probab. **28**(2), 1136–1189 (2018)

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