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Linear evolution equations in scales of Banach spaces

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ABSTRACT

This work is devoted to the study of a class of linear time-inhomogeneous evolution equations in a scale of Banach spaces. Existence, uniqueness and stability for classical solutions is provided. We study also the associated dual Cauchy problem for which we prove uniqueness in the dual scale of Banach spaces. The results are applied to an infinite system of ordinary differential equations but also to the Fokker-Planck equation associated with the spatial logistic model in the continuum.

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1. Introduction

1.1. Motivation

Interacting particle systems in the continuum can be modeled on the phase space of locally finite configurations over \mathbb{R}^d , i.e.

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$$\Gamma = \{\gamma \subset \mathbb{R}^d \mid |\gamma \cap \Lambda| < \infty \text{ for all compacts } \Lambda \subset \mathbb{R}^d\}, \quad (1.1)$$

where $|A|$ denotes the number of elements in the set $A \subset \mathbb{R}^d$. Markov dynamics on state space Γ is usually described by a (heuristic) Markov operator L acting on an appropriate class of functions $F : \Gamma \rightarrow \mathbb{R}$. In the particular case of birth-and-death evolutions it has the general form

$$(LF)(\gamma) = \sum_{x \in \gamma} d(x, \gamma \setminus \{x\})(F(\gamma \setminus \{x\}) - F(\gamma)) + \int_{\mathbb{R}^d} b(x, \gamma)(F(\gamma \cup \{x\}) - F(\gamma))dx, \quad (1.2)$$

where $d \geq 0$ is the death-rate and $b \geq 0$ the birth-rate, respectively. Other particular examples are discussed in [10], see also the references therein. In some cases, the corresponding dynamics can be obtained from stochastic differential equations [22], other examples can be treated by the theory of Dirichlet forms [1,2,30]. In most of the cases, however, one seeks to study dynamics in terms of state evolutions associated to the forward Kolmogorov equation (also called Fokker-Planck equation)

$$\frac{d}{dt} \int_{\Gamma} F(\gamma) \mu_t(d\gamma) = \int_{\Gamma} (LF)(\gamma) \mu_t(d\gamma), \quad \mu_t|_{t=0} = \mu_0, \quad (1.3)$$

where $F : \Gamma \rightarrow \mathbb{R}$ belongs to a suitable space of test functions. Note that for L of the form (1.2) one may take F from the class of polynomially bounded cylinder functions, see Section 5 for details. Since Γ is an infinite dimensional, non-linear space the study of (1.3) is a non-trivial mathematical task. Based on the work [28] the following approach for the study of the Fokker-Planck equation was established in the literature. First, we define for a reasonable class of states μ_t a sequence of correlation functions $k_{\mu_t}^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}_+$, $n \geq 0$, by the relation

$$\int_{\Gamma} \sum_{\{x_1, \dots, x_n\} \subset \gamma} f_n(x_1, \dots, x_n) \mu_t(d\gamma) = \frac{1}{n!} \int_{(\mathbb{R}^d)^n} f_n(x_1, \dots, x_n) k_{\mu_t}^{(n)}(x_1, \dots, x_n) dx_1 \dots dx_n,$$

where $f_n : (\mathbb{R}^d)^n \rightarrow \mathbb{R}_+$ is symmetric, bounded and measurable. It can be shown that each $k_{\mu_t}^{(n)}$ is symmetric and positive definite in the sense of Lenard. Moreover, there exists a one-to-one correspondence between such correlation functions and states, see, e.g., [26]. The study of (1.3) is therefore based on the study of the correlation function evolution $k_t := (k_{\mu_t}^{(n)})_{n=0}^{\infty}$ which should satisfy the Cauchy problem

$$\frac{dk_t^{(n)}}{dt} = (L^{\Delta} k_t)^{(n)}, \quad k_t^{(n)}|_{t=0} = k_0^{(n)}, \quad n \geq 0. \quad (1.4)$$

Here L^{Δ} is an operator-valued matrix acting on the components of $k_t = (k_t^{(n)})_{n=0}^{\infty}$. Note that L^{Δ} can be computed explicitly from L , see, e.g., [17] and [10]. Equation (1.4) can

be seen as a Markov analogue of the BBGKY-hierarchy known from physics. In contrast to (1.3), equation (1.4) is an infinite system of finite-dimensional evolution equations for which different functional analytic methods may be used.

Such an approach was successfully applied to various models [29,8,12,14], where the evolution of correlation functions was constructed in a weighted L^∞ -type space \mathcal{K}_α equipped with the norm

$$\|k\|_{\mathcal{K}_\alpha} = \sup_{n \geq 0} e^{-\alpha n} \|k^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)}, \quad \alpha \in \mathbb{R}.$$

A general semigroup approach is proposed in [13], where, under some suitable conditions, existence and uniqueness of classical solutions to (1.4) was shown. Since \mathcal{K}_α is a weighted L^∞ -Fock space, the main technical tool there is based on the study of the pre-dual Cauchy problem (evolution of quasi-observables) and on a clever use of sun-dual spaces for strongly continuous semigroups. It is worth noting that existence of a solution to (1.4) does not immediately yield also a solution to (1.3). It is necessary and sufficient that k_t is positive definite in the sense of Lennard, see [26] and the references therein. Such a positivity property was studied for particular models in [11,6], while a general result based on semigroup methods was obtained in [20]. As a consequence, using the theory of semigroups one is able to obtain existence of solutions to (1.3), while uniqueness holds among all solutions μ_t for which the associated sequence of correlation functions k_{μ_t} is a classical solution to (1.4). It is worth noting that the weak formulation of the Fokker-Planck equation (1.3) does not require that μ_t also provides a classical solution to (1.4). An extension of this uniqueness statement (without requiring that k_{μ_t} is a classical solution to (1.4)) was given in [20,18,19]. The latter result was essentially based on [35,36].

An application of semigroup methods often requires that the constant mortality rate $d(x, \{\emptyset\})$ is large enough, i.e. requires a ‘high-mortality regime’. Going beyond this regime, one may still construct an evolution of correlation functions obtained as the unique classical solution to (1.4) in the scale of Banach spaces $(\mathcal{K}_\alpha)_{\alpha \in \mathbb{R}}$, see, e.g., [5,16,15,25]. Existence of a solution to (1.3) is again related with the possibility to show that this classical solution is positive definite in the sense of Lennard. In such a case, one obtains a solution to (1.3) which is, in addition, unique among all solutions for which the associated correlation function evolution provides a classical solution to (1.4) in the scale $(\mathcal{K}_\alpha)_{\alpha \in \mathbb{R}}$. An abstract formulation of the methods used in this approach was recently summarized in [9].

At present there does not exist any (general) uniqueness statement for (1.3) which does not require that the correlation function evolution k_{μ_t} is a classical solution to (1.4) in the scale $(\mathcal{K}_\alpha)_\alpha$ as explained in Section 3 or in [9]. The main purpose of this work is devoted to the study of (1.3) and (1.4) with main emphasis on

- (i) Study existence and uniqueness of classical solutions in the time-inhomogeneous case.

- (ii) Stability of the solutions with respect to the initial data and the parameters of the model.
- (iii) Uniqueness for solutions to (1.3) and (1.4) without assuming that k_t is a classical solution.

Points (i) and (ii) have been already studied in [9] for the time-homogeneous case. Below we introduce an abstract formulation of this framework, and study the corresponding results in this abstract framework. An application to the spatial logistic model which corresponds to a particular case of (1.3) is discussed in Section 5.

1.2. Linear operators in a scale of Banach spaces

Let $\mathbb{E} = (\mathbb{E}_\alpha)_{\alpha > \alpha_*}$ be a scale of Banach spaces with $\alpha_* \in \mathbb{R}$ and

$$\mathbb{E}_\alpha \subset \mathbb{E}_{\alpha'}, \quad \|\cdot\|_{\alpha'} \leq \|\cdot\|_\alpha, \quad \alpha' < \alpha. \quad (1.5)$$

Here and below we let $\alpha' < \alpha$ always stand for $\alpha > \alpha' > \alpha_*$. Denote by $i_{\alpha\alpha'} \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$ the corresponding embedding operator. Here and in the following $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$ stands for the space of all bounded linear operators from \mathbb{E}_α to $\mathbb{E}_{\alpha'}$, and denote by $\|\cdot\|_{\alpha'\alpha}$ the corresponding operator norm on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$, $\alpha' < \alpha$. Let $x = y$, $x \in \mathbb{E}_{\alpha'}$, $y \in \mathbb{E}_\alpha$ stand for $i_{\alpha\alpha'}x = y$.

Definition 1.1. A bounded linear operator A in the scale \mathbb{E} is, by definition, a collection of bounded linear operators from \mathbb{E}_α to $\mathbb{E}_{\alpha'}$, i.e. $A = (A_{\alpha\alpha'})_{\alpha' < \alpha}$, $A_{\alpha\alpha'} \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$, satisfying

$$A_{\alpha''\alpha'} = i_{\alpha\alpha'} A_{\alpha''\alpha} = A_{\alpha\alpha'} i_{\alpha''\alpha}, \quad \alpha' < \alpha < \alpha''. \quad (1.6)$$

By $A \in L(\mathbb{E})$ we indicate that L is a bounded linear operator in the scale \mathbb{E} . Given two operators $A, B \in L(\mathbb{E})$, the composition $AB \in L(\mathbb{E})$ is, for all $\alpha' < \alpha$, defined by

$$(AB)_{\alpha\alpha'} := A_{\beta\alpha'} B_{\alpha\beta}, \quad (1.7)$$

where $\beta \in (\alpha', \alpha)$. It is worth noting that definition (1.7) does not depend on the particular choice of β , see (1.6). In view of (1.6) one finds that $(i_{\alpha\alpha'})_{\alpha' < \alpha} \in L(\mathbb{E})$ acts as an identity operator on $L(\mathbb{E})$. Note that such notion of linear operators includes the classical case of unbounded linear operators.

Example 1.2. For each $\alpha > \alpha_*$ let $(A_\alpha, D(A_\alpha))$ be a (possibly unbounded) linear operator in \mathbb{E}_α . Assume that $i_{\alpha\alpha'}(\mathbb{E}_\alpha) \subset D(A_{\alpha'})$ holds for all $\alpha' < \alpha$ and

$$i_{\alpha''\alpha'} A_{\alpha''} i_{\alpha\alpha''} = A_{\alpha'} i_{\alpha\alpha'} \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'}), \quad \alpha' < \alpha'' < \alpha. \quad (1.8)$$

Then $A_{\alpha\alpha'} := A_{\alpha'} i_{\alpha\alpha'}$ defines an element $A = (A_{\alpha\alpha'})_{\alpha' < \alpha} \in L(\mathbb{E})$.

1.3. Linear evolution equations in a scale of Banach spaces

Consider a family of operators $(A(t))_{t \geq 0} \subset L(\mathbb{E})$ with the property

(A1) For all $\alpha' < \alpha$, $t \mapsto A_{\alpha\alpha'}(t) \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$ is continuous in the operator norm.

We suppose that it is associated to a forward (or backward) evolution system on the scale \mathbb{E} . The precise conditions are formulated and discussed in Section 2. Let us just mention that such conditions are more general than classical evolution systems studied as studied, e.g., in [31]. They are designed, in particular, to apply to scales of weighted L^∞ -spaces, see also Section 5. Let $(B(t))_{t \geq 0} \subset L(\mathbb{E})$ be another family of operators with the properties

(B1) For all $\alpha' < \alpha$ and $k \in \mathbb{E}_\alpha$, $t \mapsto B_{\alpha\alpha'}(t)k \in \mathbb{E}_{\alpha'}$ is continuous.

(B2) There exists an increasing continuous function $M : (\alpha_*, \infty) \rightarrow [0, \infty)$ with

$$\|B_{\alpha\alpha'}(t)\|_{\alpha\alpha'} \leq \frac{M(\alpha)}{\alpha - \alpha'}, \quad t \geq 0, \quad \alpha' < \alpha. \quad (1.9)$$

We are interested in solutions to the forward and backward Cauchy problems

$$\begin{aligned} \frac{\partial u(t)}{\partial t} &= (A(t) + B(t))u(t), \quad u(s) = k, \quad t \geq s, \\ \frac{\partial v(s)}{\partial s} &= -(A(s) + B(s))v(s), \quad v(t) = k, \quad s \in [0, t] \end{aligned}$$

in the scale \mathbb{E} . The precise definitions and results are formulated in Section 3. It is worthwhile to mention that similar equations have been recently studied in the case where $B(t)$ was a nonlinear operator [32], [21]. Such equations with operators $A(t), B(t)$ as above have also applications to partial differential equations, see [34, 7, 23, 4]. We close this presentation with one simple example for which the results obtained in the subsequent sections can be applied.

Example 1.3. Let $(X, \mathcal{B}(X), \mu)$ be a σ -finite Borel space and take a family of functions $\omega_\alpha : X \rightarrow [0, \infty)$ indexed by $\alpha \in \mathbb{R}$ satisfying $\omega_{\alpha'} \leq \omega_\alpha$ for all $\alpha' < \alpha$. Define $\mathbb{E}_\alpha = L^1(X, \omega_\alpha \mu)$ with norm $\|f\|_\alpha = \int_X |f(x)| \omega_\alpha(x) \mu(dx)$. Consider the objects

(i) $m : X \rightarrow [0, \infty)$ is measurable, locally bounded and satisfies

$$\operatorname{ess\,sup}_{x \in X} m(x) \frac{\omega_{\alpha'}(x)}{\omega_\alpha(x)} < \infty, \quad \forall \alpha' < \alpha.$$

(ii) $k : X \times X \rightarrow \mathbb{R}$ is measurable and, for all $\alpha' < \alpha$, there exists an increasing function $M(\alpha) > 0$ with

$$\int_X |k(x, y)| \omega_{\alpha'}(x) \mu(dx) \leq \frac{M(\alpha)}{\alpha - \alpha'} \omega_{\alpha}(y), \quad \mu - \text{a.a. } y \in X.$$

Then A and B defined by

$$(Af)(x) := -m(x)f(x) \quad \text{and} \quad (Bf)(x) = \int_X f(y)k(x, y)\mu(dy)$$

satisfy properties (A1), (B1) and (B2). Moreover, the semigroup generated by A satisfies the conditions formulated in Section 2.

More delicate examples and applications are discussed in Sections 4 and 5.

1.4. Structure of the work

This work is organized as follows. In Section 2 we study some basic properties of forward and backward evolution systems associated to $(A(t))_{t \geq 0}$ given by (A1). The main results of this work, that is existence, uniqueness, stability and the dual Cauchy problems, are formulated and proved in Section 3. Applications to a system of ordinary differential equations and Markov evolutions in the continuum are considered in Sections 4 and 5, respectively.

2. Some simple properties for evolution systems

Let $\mathbb{E} = (\mathbb{E}_{\alpha})_{\alpha > \alpha_*}$ be a scale of Banach spaces in the sense of (1.5). Below we first introduce the notion of a forward evolution system associated to $(A(t))_{t \geq 0}$ and give some simple properties used Section 3. Afterwards we state the corresponding results (without proofs) for the backward evolution systems.

2.1. The forward evolution system

Let $(A(t))_{t \geq 0}$ be given as in (A1). Consider a family of operators $(V(t, s))_{t \geq s \geq 0}$ on the scale \mathbb{E} satisfying the following conditions

- (A2) We have $(V(t, s))_{t \geq s \geq 0} \subset L(\mathbb{E})$, and $V_{\alpha\alpha'}(t, t) = i_{\alpha\alpha'}$, for all $\alpha' < \alpha$ and $t \geq 0$. Moreover, for any $\alpha' < \alpha$ and $k \in \mathbb{E}_{\alpha}$, the mapping $(t, s) \mapsto V_{\alpha\alpha'}(t, s)k$ is continuous in $\mathbb{E}_{\alpha'}$.
- (A3) For any $\alpha' < \alpha'' < \alpha$, $k \in \mathbb{E}_{\alpha}$ and $0 \leq s \leq t$,

$$V_{\alpha\alpha'}(t, s)k = k + \int_s^t A_{\alpha''\alpha'}(r)V_{\alpha\alpha''}(r, s)kdr,$$

$$V_{\alpha\alpha'}(t, s)k = k - \int_s^t V_{\alpha''\alpha'}(t, r)A_{\alpha\alpha''}(r)kdr,$$

where the integrals exist in $\mathbb{E}_{\alpha'}$.

The following properties are immediate consequences of (A1) – (A3).

Lemma 2.1. *Let $(A(t))_{t \geq 0}$ be given as in (A1) and let $(V(t, s))_{t \geq s \geq 0}$ satisfy (A2) and (A3). Then the following assertions hold:*

- (a) *For any $\alpha' < \alpha$, $k \in \mathbb{E}_\alpha$ and $s \geq 0$, the mapping $[s, \infty) \ni t \mapsto V_{\alpha\alpha'}(t, s)k$ is continuously differentiable in $\mathbb{E}_{\alpha'}$, and we have, for all $\alpha'' \in (\alpha', \alpha)$,*

$$\frac{\partial}{\partial t} V_{\alpha\alpha'}(t, s)k = A_{\alpha''\alpha'}(t)V_{\alpha\alpha''}(t, s)k. \quad (2.1)$$

- (b) *For any $\alpha' < \alpha$, $k \in \mathbb{E}_\alpha$ and $t > 0$, the mapping $[0, t] \ni s \mapsto V_{\alpha\alpha'}(t, s)k$ is continuously differentiable in $\mathbb{E}_{\alpha'}$, and we have for all $\alpha'' \in (\alpha', \alpha)$*

$$\frac{\partial}{\partial s} V_{\alpha\alpha'}(t, s)k = -V_{\alpha''\alpha'}(t, s)A_{\alpha\alpha''}(s)k. \quad (2.2)$$

- (c) *$(V(t, s))_{t \geq s \geq 0}$ is uniquely determined by (2.1) and (2.2).*
 (d) *Suppose that, for all $\alpha' < \alpha$, the mapping $t \mapsto A_{\alpha\alpha'}(t)$ is continuous with respect to the operator norm on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$. Then assertions (a) and (b) also hold with respect to the operator norm on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$ for all $\alpha' < \alpha$.*

Proof. Using (A1) and (A2) one obtains that the integrals in (A3) are continuous in r . Hence, the left-hand sides in (A3) are continuously differentiable which yields assertions (a) and (b). Concerning (c), let $(\tilde{V}(t, s))_{t \geq s \geq 0}$ also satisfy (A2) and (A3) with the same $(A(t))_{t \geq 0}$. Take $\alpha' < \alpha_0 < \alpha_1 < \alpha$, $k \in \mathbb{E}_\alpha$ and $0 \leq s < t$, then $[s, t] \ni r \mapsto V_{\alpha_0\alpha'}(t, r)i_{\alpha_1\alpha_0}\tilde{V}_{\alpha\alpha_1}(r, s)k$ is continuously differentiable in $\mathbb{E}_{\alpha'}$ such that $\frac{d}{dr}V_{\alpha_0\alpha'}(t, r)i_{\alpha_1\alpha_0}\tilde{V}_{\alpha\alpha_1}(r, s)k = 0$. Integrating over $r \in [s, t]$ gives

$$0 = V_{\alpha''\alpha'}(t, t)\tilde{V}_{\alpha\alpha''}(t, s)k - V_{\alpha''\alpha'}(t, s)\tilde{V}_{\alpha\alpha''}(s, s)k = \tilde{V}_{\alpha\alpha'}(t, s)k - V_{\alpha\alpha'}(t, s)k,$$

where we have used $V(t, s), \tilde{V}(t, s) \in L(\mathbb{E})$ and property (A2). This proves (c). Let us prove (d). Using (A3) we obtain, for all $\alpha' < \alpha'' < \alpha$, $k \in \mathbb{E}_\alpha$ and $h \in [0, 1]$ small enough,

$$\|V_{\alpha\alpha'}(t+h, s)k - V_{\alpha\alpha'}(t, s)k\|_{\alpha'} \leq h \sup_{r \in [t, t+1]} \|A_{\alpha''\alpha'}(r)\|_{\alpha''\alpha'} \sup_{r \in [t, t+1]} \|V_{\alpha\alpha''}(r, s)\|_{\alpha\alpha''} \|k\|_\alpha,$$

and a similar estimate for $h \in [-1, 0]$ close enough to 0. Hence $t \mapsto V_{\alpha\alpha'}(t, s) \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$ is continuous in the operator norm. Similarly one can show that $s \mapsto$

$V_{\alpha\alpha'}(t, s)$ is continuous in the operator norm. From this and the continuity of $t \mapsto A_{\alpha\alpha'}(t)$ we deduce that the integrals in (A3) exist in the operator norm on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$. This proves the assertion. \square

The following remark justifies the name forward evolution system for $(V(t, s))_{t \geq s \geq 0}$.

Remark 2.2. Suppose that (A1) – (A3) are satisfied. Then, for all $t \geq r \geq s \geq 0$, it holds that $V(t, s) = V(t, r)V(r, s)$, where the composition is defined by (1.6), i.e.

$$V_{\alpha\alpha'}(t, s) = V_{\alpha''\alpha'}(t, r)V_{\alpha\alpha''}(r, s), \quad \alpha' < \alpha'' < \alpha.$$

This property can be directly deduced from previous Lemma. Since it is also a particular case of the results discussed in Section 3, we omit here the proof.

Next we provide a simple stability estimate for the forward evolution system $(V(t, s))_{t \geq s \geq 0}$.

Lemma 2.3. Let $(V(t, s))_{t \geq s \geq 0}$, $(A(t))_{t \geq 0}$ and $(\tilde{V}(t, s))_{t \geq s \geq 0}$, $(\tilde{A}(t))_{t \geq 0}$ both satisfy (A1) – (A3). Moreover, suppose that, for all $\alpha' < \alpha$, the mappings $t \mapsto A_{\alpha\alpha'}(t)$ and $t \mapsto \tilde{A}_{\alpha\alpha'}(t)$ are continuous with respect to the operator norm on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$. Then, for all $t \geq s \geq 0$ and $\alpha' < \alpha_0 < \alpha_1 < \alpha$, one has

$$\|V_{\alpha\alpha'}(t, s) - \tilde{V}_{\alpha\alpha'}(t, s)\|_{\alpha\alpha'} \leq C(\alpha', \alpha_0, \alpha_1, \alpha, t, s) \int_s^t \|A_{\alpha_1\alpha_0}(r) - \tilde{A}_{\alpha_1\alpha_0}(r)\|_{\alpha_1\alpha_0} dr,$$

where $C(\alpha', \alpha_0, \alpha_1, \alpha, t, s) = \sup_{r \in [s, t]} \|V_{\alpha_0\alpha'}(t, r)\|_{\alpha_0\alpha'} \cdot \sup_{r \in [s, t]} \|\tilde{V}_{\alpha\alpha_1}(r, s)\|_{\alpha\alpha_1}$ is finite due to (A2) and the uniform boundedness principle.

Proof. Using Lemma 2.1(d) we see that $[s, t] \ni r \mapsto V_{\alpha_0\alpha'}(t, r)i_{\alpha_1\alpha_0}\tilde{V}_{\alpha\alpha_1}(r, s)$ is continuously differentiable in $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$ and satisfies

$$\frac{d}{dr} V_{\alpha_0\alpha'}(t, r)i_{\alpha_1\alpha_0}\tilde{V}_{\alpha\alpha_1}(r, s) = V_{\alpha_0\alpha'}(t, r) \left(-A_{\alpha_1\alpha_0}(r) + \tilde{A}_{\alpha_1\alpha_0}(r) \right) \tilde{V}_{\alpha\alpha_1}(r, s).$$

Integrating over $r \in [s, t]$, using $V(t, s), \tilde{V}(t, s) \in L(\mathbb{E})$ and (A2) gives

$$\tilde{V}_{\alpha\alpha'}(t, s) - V_{\alpha\alpha'}(t, s) = \int_s^t V_{\alpha_0\alpha'}(t, r) \left(-A_{\alpha_1\alpha_0}(r) + \tilde{A}_{\alpha_1\alpha_0}(r) \right) \tilde{V}_{\alpha\alpha_1}(r, s) dr.$$

Taking the norm $\|\cdot\|_{\alpha\alpha'}$ and using the triangle inequality for the integral proves the assertion. \square

The next remark gives a sufficient condition for classical forward evolution systems to satisfy conditions (A1) – (A3), see [31,24] for their study.

Remark 2.4. For each $\alpha > \alpha_*$ let $(V_\alpha(t, s))_{t \geq s \geq 0} \subset L(\mathbb{E}_\alpha)$, and let $(A_\alpha(t), D(A_\alpha(t)))$ be linear operators on \mathbb{E}_α satisfying the following properties

- (i) $V_\alpha(t, t) = 1$, $V_\alpha(t, r)V_\alpha(r, s) = V_\alpha(t, s)$ for all $t \geq r \geq s \geq 0$ and $\alpha > \alpha_*$.
- (ii) $V_{\alpha'}(t, s)|_{\mathbb{E}_\alpha} = i_{\alpha\alpha'}V_\alpha(t, s)$ for any $\alpha' < \alpha$ and $t \geq s \geq 0$.
- (iii) $(t, s) \mapsto V_{\alpha'}(t, s)k \in \mathbb{E}_{\alpha'}$ is continuous for any $\alpha' < \alpha$ and $k \in \mathbb{E}_\alpha$.
- (iv) $i_{\alpha\alpha'}(\mathbb{E}_\alpha) \subset D(A_{\alpha'}(t))$ and (1.8) hold for all $t \geq 0$. Moreover, $[0, \infty) \ni t \mapsto A_{\alpha'}(t)i_{\alpha\alpha'} \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$ is strongly continuous for all $\alpha' < \alpha$.

Then (A1) and (A2) are satisfied by

$$V_{\alpha\alpha'}(t, s) = V_{\alpha'}(t, s)i_{\alpha\alpha'}, \quad A_{\alpha\alpha'}(t) = A_{\alpha'}(t)i_{\alpha\alpha'}, \quad \alpha' < \alpha, \quad t \geq 0.$$

Suppose, in addition, the condition

- (v) For all $\alpha' < \alpha$, $t \geq s \geq 0$ and $k \in \mathbb{E}_\alpha$,

$$V_{\alpha'}(t, s)k = k + \int_s^t A_{\alpha'}V_{\alpha'}(r, s)i_{\alpha\alpha'}kdr, \quad V_{\alpha'}(t, s)k = k - \int_s^t V_{\alpha'}(t, r)A_{\alpha'}(r)i_{\alpha\alpha'}kdr.$$

Then also property (A3) is satisfied.

In the time-homogeneous case, i.e. $A(t)$ is independent of t , above conditions can be simplified as follows.

Remark 2.5. For each $\alpha > \alpha_*$ let $(T_\alpha(t))_{t \geq 0} \subset L(\mathbb{E}_\alpha)$ be a strongly continuous semigroup with generator $(A_\alpha, D(A_\alpha))$ on \mathbb{E}_α . Suppose that

- (i) $T_{\alpha'}(t)|_{\mathbb{E}_\alpha} = T_\alpha(t)$, for all $t \geq 0$ and $\alpha' < \alpha$.
- (ii) $\mathbb{E}_\alpha \subset D(A_{\alpha'})$ and $A_{\alpha'}i_{\alpha\alpha'} \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$.

Combining properties (i) and (ii) together with [31, p. 123, Theorem 5.5] gives $A_\alpha k = A_{\alpha'}k$ for all

$$k \in D(A_\alpha) = \{h \in D(A_{\alpha'}) \cap \mathbb{E}_\alpha \mid A_{\alpha'}h \in \mathbb{E}_\alpha\} = \{h \in \mathbb{E}_\alpha \mid A_{\alpha'}h \in \mathbb{E}_\alpha\}.$$

Hence $A_{\alpha\alpha'} := A_{\alpha'}i_{\alpha\alpha'}$ and $V_{\alpha\alpha'}(t, s) := T_{\alpha\alpha'}(t - s) := T_{\alpha'}(t - s)i_{\alpha\alpha'}$ satisfy properties (A1) – (A3). Moreover, one can show that, for any $\alpha' < \alpha$ and $k \in \mathbb{E}_\alpha$, the function $t \mapsto T_{\alpha\alpha'}(t)k$ is infinitely often differentiable in $\mathbb{E}_{\alpha'}$ such that, for any $n \geq 1$,

$$\frac{d^n}{dt^n} T_{\alpha\alpha'}(t)k = A_{\alpha''\alpha'}^n T_{\alpha\alpha''}(t)k = T_{\alpha''\alpha'}(t)A_{\alpha\alpha''}^n k, \quad t \geq 0.$$

2.2. The backward evolution system

Consider a family of operators $(V(s, t))_{t \geq s \geq 0}$ on the scale \mathbb{E} satisfying the following conditions

- (A2)* We have $(V(s, t))_{t \geq s \geq 0} \subset L(\mathbb{E})$ with $V_{\alpha\alpha'}(t, t) = i_{\alpha\alpha'}$ for all $\alpha' < \alpha$ and $t \geq 0$.
 Moreover, for any $\alpha' < \alpha$ and $k \in \mathbb{E}_\alpha$, the mapping $(s, t) \mapsto V_{\alpha\alpha'}(s, t)k$ is continuous in $\mathbb{E}_{\alpha'}$.
- (A3)* For any $\alpha' < \alpha'' < \alpha$, $k \in \mathbb{E}_\alpha$ and $0 \leq s \leq t$,

$$V_{\alpha\alpha'}(s, t)k = k + \int_s^t V_{\alpha''\alpha'}(s, r)A_{\alpha\alpha''}(r)kdr,$$

$$V_{\alpha\alpha'}(s, t)k = k - \int_s^t A_{\alpha''\alpha'}(r)V_{\alpha\alpha''}(r, t)kdr.$$

The following properties are immediate consequences of (A1), and (A2)*, (A3)* and can be deduced by similar arguments to Lemma 2.1.

Lemma 2.6. *Let $(A(t))_{t \geq 0}$ be given by (A1) and $(V(s, t))_{t \geq s \geq 0}$ satisfy properties (A2)*, (A3)*. Then the following assertions hold*

- (a) *For any $\alpha' < \alpha$, $k \in \mathbb{E}_\alpha$ and $s \geq 0$, the mapping $[s, \infty) \ni t \mapsto V_{\alpha\alpha'}(s, t)k$ is continuously differentiable in $\mathbb{E}_{\alpha'}$, such that for all $\alpha'' \in (\alpha', \alpha)$,*

$$\frac{\partial}{\partial t} V_{\alpha\alpha'}(t, s)k = V_{\alpha''\alpha'}(s, t)A_{\alpha\alpha''}(t)k. \quad (2.3)$$

- (b) *For any $\alpha' < \alpha$, $k \in \mathbb{E}_\alpha$ and $t > 0$, the mapping $[0, t] \ni s \mapsto V_{\alpha\alpha'}(s, t)k$ is continuously differentiable in $\mathbb{E}_{\alpha'}$, such that for all $\alpha'' \in (\alpha', \alpha)$,*

$$\frac{\partial}{\partial s} V_{\alpha\alpha'}(t, s)k = -A_{\alpha''\alpha'}(s)V_{\alpha\alpha''}(s, t)k. \quad (2.4)$$

- (c) $(V(s, t))_{t \geq s \geq 0}$ is uniquely determined by (2.3) and (2.4).
 (d) Suppose that, for all $\alpha' < \alpha$, the mapping $t \mapsto A_{\alpha\alpha'}(t)$ is continuous with respect to the operator norm on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$. Then assertions (a) and (b) also hold with respect to the operator norm on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$ for all $\alpha' < \alpha$.

As before, the next remark explains why $(V(s, t))_{t \geq s \geq 0}$ is called backward evolution system.

Remark 2.7. Suppose that (A1), (A2)* and (A3)* are satisfied. Then $V(s, t) = V(s, r)V(r, t)$ holds for all $t \geq r \geq s \geq 0$ in the sense of (1.6).

Clearly a similar estimate to the one proved in Lemma 2.3 can be also obtained in this case.

3. Construction of the perturbation

3.1. Construction of forward evolution system

Let $(V(t, s))_{t \geq s \geq 0}$ be a family of operators as in (A2). Suppose, in addition, that

(A4) There exists a constant $K \geq 1$ such that for all $\alpha' < \alpha$

$$\|V_{\alpha\alpha'}(t, s)\|_{\alpha\alpha'} \leq K, \quad 0 \leq s \leq t.$$

Below we provide the construction of a perturbed family of operators

$$\{W_{\alpha\alpha'}(t, s) \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'}) \mid \alpha' < \alpha, \quad 0 \leq t - s < T(\alpha', \alpha)\}, \quad (3.1)$$

where $T(\alpha', \alpha) := \frac{\alpha - \alpha'}{2KeM(\alpha)}$, $\alpha' < \alpha$, with the convention that $1/0 := +\infty$. Such a construction is based on the Ovsyannikov technique, see e.g. [9] and [21] for some recent related results.

Theorem 3.1. Suppose that $(V(t, s))_{t \geq s \geq 0}$ satisfies (A2), (A4) and $(B(t))_{t \geq 0}$ satisfies (B1), (B2). Then there exists a family of operators $W(t, s)$ as in (3.1) with the properties

(a) For any $\alpha' < \alpha$ and $0 \leq t - s < T(\alpha', \alpha)$, $(t, s) \mapsto W_{\alpha\alpha'}(t, s)$ is strongly continuous on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$ with $W_{\alpha\alpha'}(t, t) = i_{\alpha\alpha'}$, and

$$\|W_{\alpha\alpha'}(t, s)\|_{\alpha\alpha'} \leq \frac{T(\alpha', \alpha)}{T(\alpha', \alpha) - (t - s)} K.$$

(b) For all $\alpha' < \alpha'' < \alpha$ with $0 \leq t - s < \min\{T(\alpha', \alpha), T(\alpha'', \alpha)\}$ we have

$$W_{\alpha\alpha'}(t, s) = i_{\alpha''\alpha'} W_{\alpha\alpha''}(t, s),$$

and for all $\alpha' < \alpha'' < \alpha$ with $0 \leq t - s < \min\{T(\alpha', \alpha), T(\alpha', \alpha'')\}$ we have

$$W_{\alpha\alpha'}(t, s) = W_{\alpha''\alpha'}(t, s) i_{\alpha\alpha''}.$$

(c) For all $\alpha' < \alpha'' < \alpha$ with $0 \leq t - s < \min\{T(\alpha', \alpha), T(\alpha'', \alpha)\}$ we have

$$W_{\alpha\alpha'}(t, s)k = V_{\alpha\alpha'}(t, s)k + \int_s^t (V(t, r)B(r))_{\alpha''\alpha'} W_{\alpha\alpha''}(r, s)k dr, \quad \forall k \in \mathbb{E}_\alpha. \quad (3.2)$$

Moreover $W(t, s)$ is unique with such property.

(d) For all $\alpha' < \alpha'' < \alpha$ with $0 \leq t - s < \min\{T(\alpha', \alpha), T(\alpha', \alpha'')\}$ we have

$$W_{\alpha\alpha'}(t, s)k = V_{\alpha\alpha'}(t, s)k + \int_s^t W_{\alpha''\alpha'}(t, r)(B(r)V(r, s))_{\alpha\alpha''}k dr, \quad \forall k \in \mathbb{E}_\alpha.$$

Moreover $W(t, s)$ is also unique with such property.

Recall (1.6) and (1.7). In order to simplify the notation in the proof, we omit the subscripts $\alpha\alpha'$, whenever no confusion may arise.

Proof. Define a sequence of operators $(W_n(t, s))_{0 \leq s \leq t} \subset L(\mathbb{E})$ by $W_0(t, s) = V(t, s)$ and

$$W_{n+1}(t, s) := \int_s^t V(t, r)B(r)W_n(r, s)dr, \quad (3.3)$$

where the integrals are, for all $\alpha' < \alpha$, defined in the strong topology on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$, while the composition $V(t, r)B(r)W_n(r, s)$ is defined by (1.7). Then, for any $\alpha' < \alpha$, $n \geq 0$ and $k \in \mathbb{E}_\alpha$, the function $W_n(t, s)k$ is continuous in $\mathbb{E}_{\alpha'}$ and satisfies

$$\|W_n(t, s)k\|_{\alpha'} \leq \|k\|_\alpha \left(\frac{t-s}{T(\alpha', \alpha)} \right)^n K.$$

Indeed, for $n = 0$ this certainly holds true due to (A4). Consider $n \geq 1$, set $\alpha_j := \alpha' + j \frac{\alpha - \alpha'}{2n}$, $j \in \{0, \dots, 2n\}$ and for $s \leq t_1 \leq \dots \leq t_n \leq t$ let

$$Q_n(t, t_1, \dots, t_n, s) := V(t, t_1)B(t_1) \cdots V(t_{2n-2}, t_{2n-1})B(t_{2n-1})V(t_{2n}, s) \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'}),$$

where the composition is again defined by (1.7). Using (1.9) and (A4), we obtain by (3.3)

$$\begin{aligned} \|W_n(t, s)k\|_{\alpha'} &\leq \int_s^t \cdots \int_s^{t_{n-1}} \|Q_n(t, t_1, \dots, t_n, s)k\|_{\alpha'} dt_n \dots dt_1 \\ &\leq K^n \|k\|_\alpha \frac{(2n)^n}{(\alpha - \alpha')^n} \int_s^t \cdots \int_s^{t_{n-1}} \prod_{j=0}^{n-1} M(\alpha_{2j+1}) dt_n \dots dt_1 \end{aligned}$$

$$\begin{aligned} &\leq \|k\|_\alpha \frac{(t-s)^n}{n!} \frac{(2M(\alpha)nK)^n}{(\alpha-\alpha')^n} \\ &\leq \|k\|_\alpha \left(\frac{t-s}{T(\alpha',\alpha)} \right)^n, \end{aligned}$$

where we have used $\frac{1}{n!} \leq \left(\frac{e}{n}\right)^n$, $n \geq 1$. Hence $\sum_{n=0}^\infty W_n(t,s)k =: W(t,s)k$ converges locally uniformly in $\mathbb{E}_{\alpha'}$ for all $0 \leq t-s < T(\alpha',\alpha)$, i.e. $(t,s) \mapsto W(t,s)k \in \mathbb{E}_{\alpha'}$ is continuous and satisfies

$$\|W(t,s)k\|_{\alpha'} \leq \|k\|_\alpha \sum_{n=0}^\infty \left(\frac{t-s}{T(\alpha',\alpha)} \right)^n K = \|k\|_\alpha \frac{T(\alpha',\alpha)}{T(\alpha',\alpha) - (t-s)} K.$$

This proves property (a). Property (b) is a direct consequence of

$$(W_n(t,s))_{\alpha'\alpha} = i_{\alpha''\alpha'}(W_n(t,s))_{\alpha\alpha''} = (W_n(t,s))_{\alpha''\alpha'} i_{\alpha\alpha''}.$$

Let us prove property (c). Take $\alpha_j := \alpha' + j \frac{\alpha-\alpha'}{2(n+1)}$, $j \in \{0, \dots, 2(n+1)\}$. Then, for $s \leq r \leq t$, we obtain

$$\begin{aligned} \|V(t,r)B(r)W_n(r,s)k\|_{\alpha'} &\leq (KM(\alpha))^{n+1} \frac{2(n+1)}{\alpha-\alpha'} \frac{(t-s)^n}{n!} \frac{(2(n+1))^n}{(\alpha-\alpha')^n} \|k\|_\alpha \\ &\leq \|k\|_\alpha \frac{4eKM(\alpha)}{\alpha-\alpha'} \left(\frac{t-s}{T(\alpha',\alpha)} \right)^n n. \end{aligned}$$

Hence the series $\sum_{n=0}^\infty V(t,r)B(r)W_n(r,s)k \in \mathbb{E}_{\alpha'}$ is locally uniformly convergent in $s \leq r \leq t$, provided one has $0 \leq t-s < T(\alpha',\alpha)$. Take $\alpha' < \alpha'' < \alpha$ with $0 \leq t-s < \min\{T(\alpha'',\alpha'), T(\alpha',\alpha)\}$. Note that for given $t-s < T(\alpha',\alpha)$ such α'' always exists. Then $W(r,s)k = \sum_{n=0}^\infty W_n(r,s)k$ converges locally uniformly in $\mathbb{E}_{\alpha''}$ and hence is also continuous in r . Since $V(t,r)B(r) \in L(\mathbb{E}_{\alpha''}, \mathbb{E}_{\alpha'})$ is strongly continuous, it follows that

$$\begin{aligned} W_{\alpha\alpha'}(t,s)k &= V_{\alpha\alpha'}(t,s)k + \sum_{n=1}^\infty \int_s^t (V(t,r)B(r))_{\alpha''\alpha'} (W_{n-1}(r,s))_{\alpha\alpha''} k dr \\ &= V_{\alpha\alpha'}(t,s)k + \int_s^t (V(t,r)B(r))_{\alpha''\alpha'} W_{\alpha\alpha''}(r,s)k dr, \end{aligned}$$

which implies that $W(t,s)$ satisfies the desired integral equation (3.2). Let us prove that $W(t,s)$ is unique with such property. Take another family of operators $\tilde{W}(t,s)$ satisfying (a), (b) and (3.2). Let $\alpha' < \alpha'' < \alpha$ with $0 \leq t-s < \min\{T(\alpha'',\alpha), T(\alpha',\alpha)\}$ and set $\alpha_j = \alpha' + j \frac{\alpha''-\alpha'}{2n}$, $j \in \{0, \dots, 2n\}$ where $n \geq 1$. Observe that, for all $s \leq t_n \leq t_{n-1} \leq \dots \leq t_1 \leq t$, one has

$$\tilde{Q}_n(t, t_1, \dots, t_n) := V(t, t_1)B(t_1) \cdots V(t_{n-1}, t_n)B(t_n) \in L(\mathbb{E}_{\alpha''}, \mathbb{E}_{\alpha'}).$$

Then, as before, we obtain

$$\|\widetilde{Q}_n(t, t_1, \dots, t_n)\|_{\alpha'\alpha''} \leq K^n \frac{M(\alpha'')^n (2n)^n}{(\alpha'' - \alpha')^n}$$

and hence we deduce that, for all $k \in \mathbb{E}_\alpha$,

$$\begin{aligned} & \|W_{\alpha\alpha'}(t, s)k - \widetilde{W}_{\alpha\alpha'}(t, s)k\|_{\alpha''} \\ & \leq \int_s^t \cdots \int_s^{t_{n-1}} \|\widetilde{Q}_n(t, t_1, \dots, t_n)\|_{\alpha''\alpha'} \|W_{\alpha\alpha''}(t_n, s)k - \widetilde{W}_{\alpha\alpha''}(t_n, s)k\|_{\alpha''} dt_n \dots dt_1 \\ & \leq C(\alpha, \alpha'', k) K^n \frac{M(\alpha'')^n (2n)^n}{(\alpha'' - \alpha')^n} \frac{(t-s)^n}{n!}, \end{aligned}$$

where $C(\alpha, \alpha'', k) = \sup_{r \in [s, t]} \|W_{\alpha\alpha''}(r, s) - \widetilde{W}_{\alpha\alpha''}(r, s)\|_{\alpha\alpha''}$ is finite by strong continuity and the uniform boundedness principle. Since the right-hand side tends to zero as $n \rightarrow \infty$, we conclude the assertion. For the last property (d), observe that the sequence $(W_n(t, s)k)_{n \in \mathbb{N}}$ also satisfies the relation

$$W_{n+1}(t, s)k = \int_s^t W_n(t, r)B(r)V(r, s)kdr.$$

A repetition of above arguments proves (d). \square

If we suppose, in addition, that also (A1) and (A3) are satisfied, then $W(t, s)$ is continuously differentiable. Consequently, we are able to solve the corresponding forward evolution equation given in the next definition.

Definition 3.2. Fix $\alpha_* < \alpha' < \alpha$, $k \in \mathbb{E}_\alpha$ and $s \geq 0$. A solution on $[s, s + T]$, $T > 0$, to the forward evolution equation

$$\frac{d}{dt} i_{\alpha'\alpha''} u(t) = (A_{\alpha'\alpha''}(t) + B_{\alpha'\alpha''}(t))u(t), \quad u(s) = k, \quad t \in [s, s + T] \quad (3.4)$$

is, by definition, a function $u \in C([s, s + T]; \mathbb{E}_{\alpha'})$, such that $i_{\alpha'\alpha''} u \in C^1([s, s + T]; \mathbb{E}_{\alpha''})$ and (3.4) holds for all $\alpha'' \in (\alpha_*, \alpha')$.

Note that the right-hand side in (3.4) is independent of the particular choice of $\alpha'' \in (\alpha_*, \alpha')$. Concerning existence and uniqueness for (3.4) we obtain the following.

Corollary 3.3. Let $(A(t))_{t \geq 0}$ be given as in (A1) and let $(V(t, s))_{t \geq s \geq 0}$ be such that (A2) – (A4) holds. Suppose that $(B(t))_{t \geq 0}$ satisfies (B1), (B2). Let $W(t, s)$ be given by Theorem 3.1. Then the following assertions hold

(a) For all $\alpha' < \alpha'' < \alpha$ and $k \in \mathbb{E}_\alpha$,

$$[s, s + \min\{T(\alpha', \alpha), T(\alpha'', \alpha)\}) \ni t \longmapsto W_{\alpha\alpha'}(t, s)k \in \mathbb{E}_{\alpha'}$$

is continuously differentiable with

$$\frac{\partial}{\partial t} W_{\alpha\alpha'}(t, s)k = (A_{\alpha''\alpha'}(t) + B_{\alpha''\alpha'}(t))W_{\alpha\alpha''}(t, s)k.$$

(b) For all $\alpha' < \alpha'' < \alpha$ and $k \in \mathbb{E}_\alpha$,

$$(t - \min\{T(\alpha', \alpha), T(\alpha', \alpha'')\}, t] \ni s \longmapsto W_{\alpha\alpha'}(t, s)k \in \mathbb{E}_{\alpha'}$$

is continuously differentiable with

$$\frac{\partial}{\partial s} W_{\alpha\alpha'}(t, s)k = -W_{\alpha''\alpha'}(t, s)(A_{\alpha\alpha''}(s) + B_{\alpha\alpha''}(s))k. \quad (3.5)$$

(c) Fix $s \geq 0$, $\alpha' < \alpha$ and $k \in \mathbb{E}_\alpha$. Suppose that there exists $T > 0$ and a function $u \in C([s, s+T]; \mathbb{E}_{\alpha'})$ which is a solution to (3.4). Then $u(t) = W_{\alpha\alpha'}(t, s)k$ holds for any t with $s \leq t < s + \min\{T, T(\alpha', \alpha)\}$.

Proof. Assertions (a) and (b) are direct consequences of Theorem 3.1(c) and Theorem 3.1(d) combined with Lemma 2.1. Let us prove (c).

Define $w(t) := W_{\alpha\alpha'}(t, s)k - u(t)$, where $s \leq t < s + \min\{T, T(\alpha', \alpha)\}$. Then $w(s) = 0$ and it suffices to show that $w = 0$. Since w solves (3.4) it follows that w satisfies, for any t with $s \leq t < s + \min\{T, T(\alpha', \alpha)\}$ and $\alpha'' \in (\alpha_*, \alpha')$,

$$i_{\alpha'\alpha''}w(t) = \int_s^t (V(t, r)B(r))_{\alpha''\alpha'}w(r)dr.$$

Iterating this equality yields

$$i_{\alpha'\alpha''}w(t) = \int_s^t \dots \int_s^{t_{n-1}} \tilde{Q}_n(t, t_1, \dots, t_n, s)_{\alpha''\alpha'}w(t_n)dt_n \dots, dt_1,$$

where $\tilde{Q}_n(t, t_1, \dots, t_n) := V(t, t_1)B(t_1) \cdots V(t_{n-1}, t_n)B(t_n) \in L(\mathbb{E}_{\alpha'}, \mathbb{E}_{\alpha''})$. In order to estimate this integral, we let $\alpha_j := \alpha'' + j \frac{\alpha' - \alpha''}{2n}$, $j \in \{0, \dots, 2n\}$. Then

$$\|\tilde{Q}_n(t, t_1, \dots, t_n)\|_{\alpha'\alpha''} \leq K^n \frac{M(\alpha')^n (2n)^n}{(\alpha' - \alpha'')^n}, \quad n \geq 1.$$

Letting $C_{\alpha'} := \sup_{r \in [s, t]} \|w(r)\|_{\alpha'} < \infty$ and using $\|w(t_n)\|_{\alpha'} \leq C_{\alpha'}$ we obtain

$$\|w(t)\|_{\alpha''} \leq C_{\alpha'} \left(\frac{2eKM(\alpha')(t-s)}{\alpha' - \alpha''} \right)^n.$$

If, in addition, $0 \leq t-s < T(\alpha'', \alpha')$, then the right-hand side tends to zero as $n \rightarrow \infty$ from which we deduce $w(t) = 0$ in $\mathbb{E}_{\alpha''}$ and thus also in \mathbb{E}_{α} for all t satisfying

$$s \leq t < s + \min \{T, T(\alpha', \alpha), T(\alpha'', \alpha')\}.$$

Setting $\alpha'' = \frac{\alpha' + \alpha_*}{2} \in (\alpha_*, \alpha')$ shows that, for any $\alpha' < \alpha$, $k \in \mathbb{E}_{\alpha}$ and $s \geq 0$, equation (3.4) has a unique solution on $[s, s + \frac{1}{2}T_0(\alpha', \alpha)]$ where $T_0(\alpha', \alpha) := \min\{T(\alpha', \alpha), \frac{\alpha' - \alpha_*}{4eKM(\alpha')}, T\}$.

Changing s to $s + \frac{1}{2}T_0(\alpha', \alpha)$ and iterating this procedure yields the assertion. Note that such an iteration is possible since the new initial condition satisfies $w(s + \frac{1}{2}T_0(\alpha', \alpha)) = 0 \in \mathbb{E}_{\alpha}$. \square

Remark 3.4. The proofs show that, if $t \mapsto B_{\alpha\alpha'}(t) \in L(\mathbb{E}_{\alpha}, \mathbb{E}_{\alpha'})$ is continuous in the operator norm for any $\alpha' < \alpha$, then $W_{\alpha\alpha'}(t, s)$ is continuously differentiable in the operator norm on $L(\mathbb{E}_{\alpha}, \mathbb{E}_{\alpha'})$.

As a consequence of the uniqueness results we see that $W(t, s)$ satisfies the forward evolution property.

Corollary 3.5. Let $(A(t))_{t \geq 0}$ be given as in (A1) and let $(V(t, s))_{t \geq s \geq 0}$ be such that (A2) – (A4) holds. Suppose that $(B(t))_{t \geq 0}$ satisfies (B1), (B2). Let $W(t, s)$ be the evolution system given by Theorem 3.1. Then, for all $\alpha' < \alpha'' < \alpha$ and $0 \leq s \leq r \leq t$ satisfying the relations

$$t - s < T(\alpha', \alpha), \quad r - s < T(\alpha'', \alpha), \quad t - r < T(\alpha', \alpha''),$$

it holds that $W_{\alpha\alpha'}(t, s) = W_{\alpha''\alpha'}(t, r)W_{\alpha\alpha''}(r, s)$.

Remark 3.6. If $B(t) = 0$ for all $t \geq 0$, then $M(\alpha) = 0$ and hence $T(\alpha', \alpha) = +\infty$. Consequently $W_{\alpha\alpha'}(t, s) = V_{\alpha\alpha'}(t, s)$ is defined for all $t \geq s \geq 0$.

Below we provide one sufficient condition under which $W(t, s)$ is defined for all $t \geq s \geq 0$.

Remark 3.7. Suppose that the same conditions as for Theorem 3.1 are satisfied, and assume that

$$\sup_{\alpha > \alpha_*} M(\alpha) =: M^* < \infty.$$

Let $W(t, s)$ be the evolution system given by Theorem 3.1 with $T(\alpha', \alpha) = \frac{\alpha - \alpha'}{2eKM(\alpha)}$, and let $W^0(t, s)$ be the evolution system given by Theorem 3.1 with $T_0(\alpha', \alpha) = \frac{\alpha - \alpha'}{2eKM^*}$. By uniqueness we obtain, for all $\alpha' < \alpha$,

$$W_{\alpha\alpha'}(t, s) = W_{\alpha\alpha'}^0(t, s), \quad 0 \leq t - s < T_0(\alpha', \alpha) \leq T(\alpha', \alpha).$$

Moreover, for each $T > 0$ and $\alpha' > \alpha_*$ there exists $\alpha = \alpha(T) > \alpha'$ such that $T < T_0(\alpha', \alpha)$, i.e. $W_{\alpha\alpha'}(t, s)$ is defined on $[0, T]$. This operator is clearly independent of the particular choice of $\alpha(T)$ as long as $T < T_0(\alpha', \alpha)$, see also Theorem 3.1.(b).

3.2. Stability for the forward evolution system

In this section we study stability for the constructed evolution systems. Consider, for each $n \in \mathbb{N}$, the following set of conditions

- (S1) Let $(A^{(n)}(t))_{t \geq 0} \subset L(\mathbb{E})$ be such that $t \mapsto A_{\alpha\alpha'}^{(n)}(t) \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$, $\alpha' < \alpha$, is continuous in the operator topology.
- (S2) Let $(V^{(n)}(t, s))_{t \geq s \geq 0} \subset L(\mathbb{E})$ satisfy (A2) and (A3) for the operators $(A^{(n)}(t))_{t \geq 0}$. Moreover, suppose that there exists a constant $K \geq 1$ such that

$$\|V_{\alpha\alpha'}^{(n)}(t, s)\|_{\alpha\alpha'} \leq K, \quad 0 \leq s \leq t, \quad \alpha' < \alpha, \quad n \in \mathbb{N}. \quad (3.6)$$

- (S3) Let $(B^{(n)}(t))_{t \geq 0} \subset L(\mathbb{E})$ be such that $t \mapsto B_{\alpha\alpha'}^{(n)}(t) \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$, $\alpha' < \alpha$, is strongly continuous. Moreover, suppose that there exists an increasing continuous function $M : (\alpha_*, \infty) \rightarrow [0, \infty)$ independent of n such that

$$\|B_{\alpha\alpha'}^{(n)}(t)\|_{\alpha\alpha'} \leq \frac{M(\alpha)}{\alpha - \alpha'}, \quad \alpha' < \alpha, \quad t \geq 0. \quad (3.7)$$

Then we obtain the following.

Theorem 3.8. Suppose that (S1) – (S3) are satisfied. Let $(A(t))_{t \geq 0}$ and $(B(t))_{t \geq 0}$ be two families of operators in $L(\mathbb{E})$ such that, for any $T > 0$ and $\alpha' < \alpha$,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|B_{\alpha\alpha'}^{(n)}(t) - B_{\alpha\alpha'}(t)\|_{\alpha\alpha'} = 0 \quad (3.8)$$

and

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|A_{\alpha\alpha'}^{(n)}(t) - A_{\alpha\alpha'}(t)\|_{\alpha\alpha'} = 0. \quad (3.9)$$

Then $(A(t))_{t \geq 0}$ satisfies (A1), $(B(t))_{t \geq 0}$ satisfies (B1) and (B2) with $M(\alpha)$ being the same as in (3.7), and there exists $(V(t, s))_{t \geq s \geq 0}$ satisfying (A2) – (A4) with a constant

K being the same as in (3.6). Moreover, for any $n \in \mathbb{N}$, there exist operators $W^{(n)}(t, s)$ and $W(t, s)$ given by Theorem 3.1 with $T(\alpha', \alpha) = \frac{\alpha - \alpha'}{2eKM(\alpha)}$ and, for any $\alpha' < \alpha$ and any compact $\Delta_{\alpha'\alpha} \subset \{(t, s) \mid 0 \leq t - s < T(\alpha', \alpha)\}$, it holds that

$$\lim_{n \rightarrow \infty} \sup_{(t,s) \in \Delta_{\alpha'\alpha}} \|W_{\alpha\alpha'}^{(n)}(t, s)k - W_{\alpha\alpha'}(t, s)k\|_{\alpha'} = 0, \quad k \in \mathbb{E}_\alpha. \quad (3.10)$$

Proof. From (3.9) it follows that $t \mapsto A_{\alpha\alpha'}(t)$ is continuous in the operator norm and hence satisfies (A1). Analogously, one shows that (3.8) implies (B1) and (B2). Applying Lemma 2.3 to $V^{(n)}(t, s)$ and $V^{(m)}(t, s)$ yields, for all $t \geq s \geq 0$ and $\alpha' < \alpha_0 < \alpha_1 < \alpha$,

$$\|V_{\alpha\alpha'}^{(n)}(t, s) - \tilde{V}_{\alpha\alpha'}^{(m)}(t, s)\|_{\alpha\alpha'} \leq C(\alpha_0, \alpha_1, \alpha', \alpha)K^2 \int_s^t \|A_{\alpha_1\alpha_0}^{(n)}(r) - A_{\alpha_1\alpha_0}^{(m)}(r)\|_{\alpha_1\alpha_0} dr.$$

Hence $(V_{\alpha\alpha'}^{(n)}(t, s))_{n \in \mathbb{N}}$ is a Cauchy sequence in $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$. Denote by $V_{\alpha\alpha'}(t, s)$ its limit. Then

$$\lim_{n \rightarrow \infty} \sup_{(t,s) \in \Delta_{\alpha'\alpha}} \|V_{\alpha\alpha'}^{(n)}(t, s) - V_{\alpha\alpha'}(t, s)\|_{\alpha\alpha'} = 0. \quad (3.11)$$

It is not difficult to show that $(V(t, s))_{t \geq s \geq 0}$ satisfies the properties (A2) – (A4). Hence we may apply Theorem 3.1 and obtain the existence of $W^{(n)}(t, s), W(t, s)$ given by

$$W(t, s) = \sum_{j=0}^{\infty} W_j(t, s), \quad W^{(n)}(t, s) = \sum_{j=0}^{\infty} W_j^{(n)}(t, s), \quad (3.12)$$

where $W_j(t, s)$ and $W_j^{(n)}(t, s)$ are defined in the proof of Theorem 3.1. Arguing as in the proof of Theorem 3.1, we obtain from (3.6) and (3.7) the inequality

$$\|W_j(t, s)\|_{\alpha\alpha'}, \quad \|W_j^{(n)}(t, s)\|_{\alpha\alpha'} \leq \left(\frac{t-s}{T(\alpha', \alpha)} \right)^j K, \quad n \geq 1, \quad j \geq 0.$$

This implies that the series in (3.12) converges in the strong operator topology on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$, $\alpha' < \alpha$, uniformly with respect to $(t, s) \in \Delta_{\alpha'\alpha}$ and $n \in \mathbb{N}$. Thus it suffices to show that, for all $j \geq 0$ and $\alpha' < \alpha$,

$$\lim_{n \rightarrow \infty} \sup_{(t,s) \in \Delta_{\alpha'\alpha}} \|W_j^{(n)}(t, s)k - W_j(t, s)k\|_{\alpha'} = 0, \quad k \in \mathbb{E}_\alpha.$$

For $j = 0$ this follows from (3.11). For $j \geq 1$ we proceed by induction using the definition of $W_j^{(n)}(t, s), W_j(t, s)$ and (3.8). \square

Remark 3.9. If, in addition to the conditions of Theorem 3.8, the mapping

$$[0, \infty) \ni t \mapsto B_{\alpha\alpha'}^{(n)}(t) \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$$

is continuous in the operator norm for all $\alpha' < \alpha$. Then (3.12) converges in the operator norm and hence (3.10) also holds in the operator norm. This includes e.g. the time-homogeneous case.

3.3. Construction of backward evolution system

In this section we give an analogous construction for the corresponding backward evolution system. Namely, we consider a family of operators $(V(s, t))_{t \geq s \geq 0}$ with the property (A2)* and we assume that

(A4)* There exists a constant $K \geq 1$ such that for all $\alpha' < \alpha$

$$\|V_{\alpha\alpha'}(s, t)\|_{\alpha\alpha'} \leq K, \quad 0 \leq s \leq t.$$

The next statement is proved analogously to Theorem 3.1.

Theorem 3.10. Suppose that $(V(s, t))_{t \geq s \geq 0}$ satisfies (A2)*, (A4)* and $(B(t))_{t \geq 0}$ satisfies (B1), (B2). Define $T(\alpha', \alpha) := \frac{\alpha - \alpha'}{2KeM(\alpha)}$, $\alpha' < \alpha$. Then there exists a family of operators

$$\{W_{\alpha\alpha'}(s, t) \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'}) \mid \alpha' < \alpha, \quad 0 \leq t - s < T(\alpha', \alpha)\}$$

with the properties

(a) For any $\alpha' < \alpha$ and $0 \leq t - s < T(\alpha', \alpha)$ we have $W_{\alpha\alpha'}(s, t) \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$, $(s, t) \mapsto W_{\alpha\alpha'}(s, t)$ is strongly continuous on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$ with $W_{\alpha\alpha'}(t, t) = i_{\alpha\alpha'}$, and

$$\|W_{\alpha\alpha'}(s, t)\|_{\alpha\alpha'} \leq \frac{T(\alpha', \alpha)}{T(\alpha', \alpha) - (t - s)} K.$$

(b) For all $\alpha' < \alpha'' < \alpha$ with $0 \leq t - s < \min\{T(\alpha', \alpha), T(\alpha'', \alpha)\}$ we have

$$W_{\alpha\alpha'}(s, t) = i_{\alpha''\alpha'} W_{\alpha\alpha''}(s, t),$$

and for all $\alpha' < \alpha'' < \alpha$ with $0 \leq t - s < \min\{T(\alpha', \alpha), T(\alpha', \alpha'')\}$ we have

$$W_{\alpha\alpha'}(s, t) = W_{\alpha''\alpha'}(s, t) i_{\alpha\alpha''}.$$

(c) For all $\alpha' < \alpha'' < \alpha$ with $0 \leq t - s < \min\{T(\alpha', \alpha), T(\alpha', \alpha'')\}$ we have

$$W_{\alpha\alpha'}(s, t)k = V_{\alpha\alpha'}(s, t)k + \int_s^t W_{\alpha''\alpha'}(s, r)(B(r)V(r, t))_{\alpha\alpha''}k dr, \quad \forall k \in \mathbb{E}_\alpha.$$

Moreover $W(s, t)$ is unique with such property.

(d) For all $\alpha' < \alpha'' < \alpha$ with $0 \leq t - s < \min\{T(\alpha', \alpha), T(\alpha'', \alpha)\}$ we have

$$W_{\alpha\alpha'}(s, t)k = V_{\alpha\alpha'}(s, t)k + \int_s^t (V(s, r)B(r))_{\alpha''\alpha'}W_{\alpha\alpha''}(r, t)k dr, \quad \forall k \in \mathbb{E}_\alpha.$$

Moreover $W(s, t)$ is also unique with such property.

Proof. Since the proof is very similar to Theorem 3.1, we only sketch the main differences. Define a sequence of operators $(W_n(s, t))_{0 \leq s \leq t} \subset L(\mathbb{E})$ by $W_0(s, t) = V(s, t)$ and

$$W_{n+1}(s, t) := \int_s^t W_n(s, r)B(r)V(r, t)dr,$$

where the integrals are, for all $\alpha' < \alpha$, defined in the strong topology on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$. As before we can show that, for any $\alpha' < \alpha$, $n \geq 0$ and $k \in \mathbb{E}_\alpha$, the function $W_n(s, t)k \in \mathbb{E}_{\alpha'}$ is continuous in (s, t) and satisfies

$$\|W_n(s, t)k\|_{\alpha'} \leq \|k\|_\alpha \left(\frac{t-s}{T(\alpha', \alpha)} \right)^n K.$$

From this we readily deduce assertion (a). Assertion (b) is again a direct consequence of

$$(W_n(s, t))_{\alpha'\alpha} = i_{\alpha''\alpha'}(W_n(s, t))_{\alpha\alpha''} = (W_n(s, t))_{\alpha''\alpha'}i_{\alpha\alpha''}.$$

Assertion (c) can be shown in the same way as Theorem 3.1.(c). For the last property (d), observe that the sequence $(W_n(s, t))_{n \in \mathbb{N}}$ also satisfies the relation

$$W_{n+1}(s, t) = \int_s^t V(s, r)B(r)W_n(r, t)dr$$

from which we may deduce (d). \square

Below we relate this evolution family $W(s, t)$ with a backward Cauchy problem.

Definition 3.11. Fix $\alpha_* < \alpha' < \alpha$, $k \in \mathbb{E}_\alpha$ and $t \geq 0$. A solution on $[t - T, t]$, $T \in (0, t]$, to the backward Cauchy problem

$$\frac{d}{ds} i_{\alpha'\alpha''} v(s) = -(A_{\alpha'\alpha''}(s) + B_{\alpha'\alpha''}(s))v(s), \quad v(t) = k, \quad s \in [t - T, t] \quad (3.13)$$

is, by definition, a function $v \in C([0, t]; \mathbb{E}_{\alpha'})$, such that $i_{\alpha'\alpha''} u \in C^1([t - T, t]; \mathbb{E}_{\alpha''})$ and (3.13) holds for all $\alpha'' \in (\alpha_*, \alpha')$.

Corollary 3.12. Let $(A(t))_{t \geq 0}$ be given as in (A1) and let $(V(s, t))_{t \geq s \geq 0}$ be such that $(A2)^* - (A4)^*$ holds. Suppose that $(B(t))_{t \geq 0}$ satisfies (B1), (B2). Let $W(s, t)$ be the evolution system given by Theorem 3.10. Then the following assertions hold

(a) For all $\alpha' < \alpha'' < \alpha$ and $k \in \mathbb{E}_\alpha$,

$$[s, s + \min\{T(\alpha', \alpha), T(\alpha', \alpha'')\}] \ni t \longmapsto W_{\alpha\alpha'}(s, t)k \in \mathbb{E}_{\alpha'}$$

is continuously differentiable with

$$\frac{\partial}{\partial t} W_{\alpha\alpha'}(s, t)k = W_{\alpha''\alpha'}(s, t)(A_{\alpha\alpha''}(t) + B_{\alpha\alpha''}(t))k.$$

(b) For all $\alpha' < \alpha'' < \alpha$ and $k \in \mathbb{E}_\alpha$,

$$(t - \min\{T(\alpha', \alpha), T(\alpha'', \alpha)\}, t] \ni s \longmapsto W_{\alpha\alpha'}(s, t)k \in \mathbb{E}_{\alpha'}$$

is continuously differentiable with

$$\frac{\partial}{\partial s} W_{\alpha\alpha'}(s, t)k = -(A_{\alpha''\alpha'}(s) + B_{\alpha''\alpha'}(s))W_{\alpha\alpha'}(s, t)k.$$

(c) Fix $t > 0$, $\alpha' < \alpha$ and $k \in \mathbb{E}_\alpha$. Let u be a solution to (3.13) on $[T - t, t]$, where $T \in (0, t]$. Then $u(s) = W_{\alpha\alpha'}(s, t)k$ holds for any s with $t - \min\{T, T(\alpha', \alpha)\} < s \leq t$.

Proof. Follows by similar arguments to Corollary 3.3. \square

Remark 3.13. The proofs show that, if $t \longmapsto B_{\alpha\alpha'}(t) \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$ is continuous in the operator norm for any $\alpha' < \alpha$, then $W_{\alpha\alpha'}(s, t)$ is continuously differentiable in the operator norm on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$.

As a consequence we can deduce the backward evolution property for $W(s, t)$.

Corollary 3.14. Let $(A(t))_{t \geq 0}$ be given as in (A1) and let $(V(s, t))_{t \geq s \geq 0}$ be such that $(A2)^* - (A4)^*$ holds. Suppose that $(B(t))_{t \geq 0}$ satisfies (B1), (B2). Let $W(s, t)$ be the

evolution system given by Theorem 3.10. Then, for all $\alpha' < \alpha'' < \alpha$ and $0 \leq s \leq r \leq t$ satisfying the relations

$$t - s < T(\alpha', \alpha), \quad r - s < T(\alpha'', \alpha), \quad t - r < T(\alpha', \alpha''),$$

it holds that $W_{\alpha\alpha'}(s, t) = W_{\alpha''\alpha'}(s, r)W_{\alpha\alpha''}(r, t)$.

Remark 3.15. A similar stability result as for the forward evolution system can be also obtained in this case.

3.4. The dual Cauchy problem

Set $\mathbb{B}_\alpha := \mathbb{E}_\alpha^*$ with $\|\cdot\|_{\mathbb{B}_\alpha} =: \|\cdot\|_\alpha$, for $\alpha > \alpha_*$. Then, for any $\alpha' < \alpha$, $\mathbb{B}_{\alpha'} \ni \ell \mapsto \ell|_{\mathbb{E}_\alpha} \in \mathbb{B}_\alpha$ defines an embedding with $\|\ell|_{\mathbb{E}_\alpha}\|_\alpha \leq \|\ell\|_{\alpha'}$. By abuse of notation, we denote this embedding also by $i_{\alpha'\alpha}$. Hence $\mathbb{B} = (\mathbb{B}_\alpha)_{\alpha > \alpha_*}$ is a scale of Banach spaces satisfying

$$\mathbb{B}_{\alpha'} \subset \mathbb{B}_\alpha, \quad \|\cdot\|_\alpha \leq \|\cdot\|_{\alpha'}, \quad \alpha' < \alpha. \quad (3.14)$$

Bounded linear operators in the scale \mathbb{B} , the space $L(\mathbb{B})$ and the composition of bounded linear operators on \mathbb{B} , are defined analogously to the scale \mathbb{E} . Denote by $\langle k, \ell \rangle_\alpha := \ell(k)$, where $k \in \mathbb{E}_\alpha$, $\ell \in \mathbb{B}_\alpha$ and $\alpha > \alpha_*$, the dual pairing between \mathbb{E}_α and \mathbb{B}_α . Then, for all $\alpha' < \alpha$,

$$\langle i_{\alpha\alpha'}k, \ell \rangle_{\alpha'} = \langle k, i_{\alpha'\alpha}\ell \rangle_\alpha, \quad k \in \mathbb{E}_\alpha, \quad \ell \in \mathbb{B}_{\alpha'}.$$

We omit the subscript α , if no confusion may arise. For given $Q \in L(\mathbb{E})$ the adjoint operator $Q^* \in L(\mathbb{B})$ is defined by

$$(Q_{\alpha'\alpha}^*\ell)(k) := \ell(Q_{\alpha\alpha'}k), \quad k \in \mathbb{E}_\alpha, \quad \ell \in \mathbb{B}_{\alpha'}, \quad \alpha' < \alpha,$$

and hence satisfies $\langle Q_{\alpha\alpha'}k, \ell \rangle = \langle k, Q_{\alpha'\alpha}^*\ell \rangle$ for $k \in \mathbb{E}_\alpha$ and $\ell \in \mathbb{B}_{\alpha'}$.

Definition 3.16. Let $(A(t))_{t \geq 0}$ be given as in (A1) and let $(B(t))_{t \geq 0}$ satisfy (B1) and (B2). Let $\mathcal{Y} \subset \bigcap_{\beta > \alpha_*} \mathbb{E}_\beta$ be such that \mathcal{Y} is dense in \mathbb{E}_β for each $\beta > \alpha_*$. Fix $\alpha' < \alpha$ and $\ell \in \mathbb{B}_{\alpha'}$.

(a) Fix $t > 0$ and $T \in (0, t]$. A solution to

$$\langle k, \ell(s) \rangle = \langle k, \ell \rangle + \int_s^t \langle (A(r) + B(r))_{\alpha'\alpha} k, \ell(r) \rangle dr, \quad s \in [T - t, t] \quad (3.15)$$

is, by definition, a family $(\ell(s))_{s \in [T-t, t]} \subset \mathbb{B}_\alpha$ such that

- (i) $[T - t, t] \ni s \mapsto \langle k, \ell(s) \rangle$ is continuous for any $k \in \mathbb{E}_\alpha$.
(ii) (3.15) holds for any $k \in \mathcal{Y}$ and all $\alpha'' > \alpha$.
(b) Fix $s \geq 0$ and $T > 0$. A solution to

$$\langle k, \ell(t) \rangle = \langle k, \ell \rangle + \int_s^t \langle (A(r) + B(r))_{\alpha''\alpha} k, \ell(r) \rangle dr, \quad t \in [s, s + T] \quad (3.16)$$

is, by definition, a family $(\ell(t))_{t \in [s, s+T]} \subset \mathbb{B}_\alpha$ such that

- (i) $[s, s + T] \ni t \mapsto \langle k, \ell(t) \rangle$ is continuous for any $k \in \mathbb{E}_\alpha$.
(ii) (3.16) holds for any $k \in \mathcal{Y}$ and all $\alpha'' > \alpha$.

Note that, as before, the right-hand sides in (3.15) and (3.16) are independent of the particular choice of $\alpha'' > \alpha$. The following is our main result for the dual Cauchy problems.

Theorem 3.17. *Let $(A(t))_{t \geq 0}$ be given as in (A1) and let $(B(t))_{t \geq 0}$ satisfy (B1) and (B2). Let $\mathcal{Y} \subset \bigcap_{\beta > \alpha_*} \mathbb{E}_\beta$ be such that \mathcal{Y} is dense in \mathbb{E}_α is dense for any $\alpha > \alpha_*$. Fix $\alpha' < \alpha$ and $\ell \in \mathbb{B}_{\alpha'}$. Then the following assertions hold:*

- (a) *Let $t > 0$, $T \in (0, t]$, and suppose that (A2) – (A4) are satisfied. Then each solution $(\ell(s))_{t-T \leq s \leq t} \subset \mathbb{B}_\alpha$ to (3.15) satisfies*

$$\ell(s) = W_{\alpha'\alpha}(t, s)^* \ell, \quad \forall s \text{ with } t - \min\{T, T(\alpha', \alpha)\} < s \leq t.$$

Moreover, for all $k \in \mathbb{E}_\alpha$ and $\alpha'' \in (\alpha', \alpha)$, the function

$$(t - \min\{T(\alpha', \alpha), T(\alpha', \alpha'')\}, t] \ni s \mapsto \langle k, W_{\alpha'\alpha}(t, s)^* \ell \rangle$$

is continuously differentiable with

$$\frac{d}{ds} \langle k, W_{\alpha'\alpha}(t, s)^* \ell \rangle = -\langle (A_{\alpha''\alpha}(s) + B_{\alpha\alpha''}(s))k, W_{\alpha'\alpha''}(t, s)^* \ell \rangle.$$

- (b) *Let $s \geq 0$ and $T > 0$. Suppose that (A2)* – (A4)* are satisfied. Then each solution $(\ell(t))_{t \in [s, s+T]} \subset \mathbb{B}_\alpha$ to (3.16) satisfies*

$$\ell(t) = W_{\alpha'\alpha}(s, t)^* \ell, \quad \forall t \text{ with } s \leq t < s + \min\{T, T(\alpha', \alpha)\}.$$

Moreover, for all $k \in \mathbb{E}_\alpha$ and $\alpha'' \in (\alpha', \alpha)$, the function

$$[s, s + \min\{T(\alpha', \alpha), T(\alpha', \alpha'')\}) \ni t \mapsto \langle k, W_{\alpha'\alpha}(s, t)^* \ell \rangle$$

is continuously differentiable with

$$\frac{d}{dt} \langle k, W_{\alpha' \alpha}(s, t)^* \ell \rangle = \langle (A_{\alpha'' \alpha}(t) + B_{\alpha'' \alpha}(t))k, W_{\alpha' \alpha''}(s, t)^* \ell \rangle.$$

Proof. We will prove only assertion (a). Assertion (b) can be deduced by similar arguments. It is clear that $\ell(s)$ given by $\ell(s) = W_{\alpha' \alpha}(t, s)^* \ell$ is, by duality and Corollary 3.3, a solution to (3.15). Let us show that it is the only solution. Take any solution $(\ell(s))_{t-T \leq s \leq t} \subset \mathbb{B}_\alpha$ to (3.15). Then, for all $\alpha'' > \alpha$ and $k \in \mathbb{E}_{\alpha''}$, $s \mapsto \langle i_{\alpha'' \alpha} k, \ell(s) \rangle$ is continuously differentiable with

$$\frac{d}{ds} \langle i_{\alpha'' \alpha} k, \ell(s) \rangle = -\langle (A_{\alpha'' \alpha}(s) + B_{\alpha'' \alpha}(s))k, \ell(s) \rangle. \quad (3.17)$$

Indeed, let $(k_n)_{n \in \mathbb{N}} \subset \mathcal{Y}$ be such that $\|k_n - k\|_{\alpha''} \rightarrow 0$ as $n \rightarrow \infty$. Then

$$\langle i_{\alpha'' \alpha} k_n, \ell(s) \rangle = \langle i_{\alpha'' \alpha} k_n, \ell \rangle + \int_s^t \langle (A_{\alpha'' \alpha}(r) + B_{\alpha'' \alpha}(r))_{\alpha'' \alpha} k_n, \ell(r) \rangle dr, \quad n \geq 1$$

and one has

$$\|(A_{\alpha'' \alpha}(r) + B_{\alpha'' \alpha}(r))_{\alpha'' \alpha} k_n\|_\alpha \leq \sup_{n \geq 1} \|k_n\|_{\alpha''} \sup_{r \in [s, t]} \|(A_{\alpha'' \alpha}(r) + B_{\alpha'' \alpha}(r))_{\alpha'' \alpha}\|_{\alpha'' \alpha} < \infty.$$

Since $(A_{\alpha'' \alpha}(r) + B_{\alpha'' \alpha}(r))_{\alpha'' \alpha} \in L(\mathbb{E}_{\alpha''}, \mathbb{E}_\alpha)$, we can take the limit $n \rightarrow \infty$ to deduce that, for any $s \in [t - T, t]$,

$$\langle i_{\alpha'' \alpha} k, \ell(s) \rangle = \langle i_{\alpha'' \alpha} k, \ell \rangle + \int_s^t \langle (A_{\alpha'' \alpha}(r) + B_{\alpha'' \alpha}(r))_{\alpha'' \alpha} k, \ell(r) \rangle dr,$$

i.e. (3.17) is satisfied.

Define $w(s) := \ell(s) - W_{\alpha' \alpha}(t, s)^* \ell$. Then, for all $\alpha'' > \alpha$ and $k \in \mathbb{E}_{\alpha''}$, it follows that

$$\begin{aligned} \langle k, w(s) \rangle &= \int_s^t \langle (A(r) + B(r))_{\alpha'' \alpha} k, w(r) \rangle dr \\ &= \int_s^t \cdots \int_s^{t_{n-1}} \langle Q_n(t, t_1, \dots, t_n)_{\alpha'' \alpha} k, w(r) \rangle dt_n \cdots dt_1, \end{aligned}$$

where $Q_n(t, t_1, \dots, t_n)_{\alpha'' \alpha} = ((A(t_n) + B(t_n)) \cdots (A(t_1) + B(t_1)))_{\alpha'' \alpha}$ can be estimated in the same way as in Theorem 3.1, i.e.

$$\|Q_n(t, t_1, \dots, t_n)_{\alpha'' \alpha}\|_{\alpha'' \alpha} \leq \left(\frac{2nKM(\alpha'')}{\alpha'' - \alpha} \right)^n, \quad n \geq 1.$$

Hence we obtain

$$\begin{aligned}
|\langle k, w(s) \rangle| &\leq \int_s^t \cdots \int_s^{t_{n-1}} \|Q_n(t, t_1, \dots, t_n)_{\alpha''\alpha} \|_{\alpha''\alpha} \|k\|_{\alpha''} \sup_{r \in [s, t]} \|w(r)\|_{\alpha} dt_n \dots dt_1 \\
&\leq \|k\|_{\alpha''} \sup_{r \in [s, t]} \|w(r)\|_{\alpha} \left(\frac{2nKM(\alpha'')}{\alpha'' - \alpha} \right)^n \frac{(t-s)^n}{n!}.
\end{aligned}$$

Since for $0 \leq t-s < \min\{T, T(\alpha', \alpha), T(\alpha, \alpha'')\}$ the right-hand side tends to zero as $n \rightarrow \infty$, we conclude that $w(s) = 0$. Letting $\alpha'' = \alpha + 1$ yields for $T_0(\alpha', \alpha) = \min\left\{T, T(\alpha', \alpha), \frac{1}{2eKM(\alpha+1)}\right\}$

$$w(s) = 0, \quad t - \frac{1}{2}T_0(\alpha', \alpha) \leq s \leq t.$$

Since $w(t - \frac{1}{2}T_0(\alpha', \alpha)) = 0 \in \mathbb{B}_{\alpha'}$, we may iterate this argument by changing $t \mapsto t - \frac{1}{2}T_0(\alpha', \alpha)$. This proves the assertion. \square

Let us close this section with the example considered before.

Remark 3.18. Let $\mathbb{E} = (\mathbb{E}_{\alpha})_{\alpha > \alpha_*}$ be a scale of Banach spaces and let, for each $\alpha > \alpha_*$, $(T_{\alpha}(t))_{t \geq 0}$ be a strongly continuous semigroup with generator $(A_{\alpha}, D(A_{\alpha}))$ satisfying properties (i) and (ii) of Remark 2.5. Let $B(t)$ be given with properties (B1) and (B2). Then all results of this section are applicable to this case.

4. Infinite system of ordinary differential equations

Let $(a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix having complex-valued entries and $x = (x_n)_{n=0}^{\infty}$ be the initial condition. We apply our results to the infinite system of ordinary differential equations

$$\frac{du_n(t)}{dt} = \sum_{k=0}^{\infty} a_{nk} u_k(t), \quad u_n(0) = x_n, \quad n \in \mathbb{N}_0, \quad (4.1)$$

where $u(t) = (u_n(t))_{n=0}^{\infty}$ is a sequence of complex numbers. For $\alpha \in \mathbb{R}$ let

$$\mathbb{E}_{\alpha} := \left\{ u = (u_n)_{n=0}^{\infty} \mid \|u\|_{\alpha} := \sum_{n=0}^{\infty} |u_n| e^{\alpha n} < \infty \right\},$$

be the Banach space of all complex-valued sequences with norm $\|\cdot\|_{\alpha}$. Then $\mathbb{E} = (\mathbb{E}_{\alpha})_{\alpha}$ defines a scale of Banach spaces in the sense of (1.5).

Remark 4.1. Let B be a linear mapping given by

$$(Bu)_n := \sum_{k=0}^{\infty} b_{nk} u_k, \quad n \in \mathbb{N}_0,$$

provided $u = (u_n)_{n=0}^\infty$ is such that the series is absolutely convergent. Then $B \in L(\mathbb{E})$ iff one has

$$\sup_{k \geq 0} e^{-\alpha k} \sum_{n=0}^{\infty} |b_{nk}| e^{\alpha' n} < \infty, \quad \alpha' < \alpha.$$

In such a case it holds that $B_{\alpha\alpha'} u = Bu$ and $\|B_{\alpha\alpha'}\| = \sup_{k \geq 0} e^{-\alpha k} \sum_{n=0}^{\infty} |b_{nk}| e^{\alpha' n}$.

We study (4.1) under the following conditions

(E1) There exist $(d_n)_{n \in \mathbb{N}_0} \subset (0, \infty)$ and $(b_{nk})_{n,k \in \mathbb{N}_0}, (c_{nk})_{n,k \in \mathbb{N}_0} \subset \mathbb{C}$ such that

$$a_{nk} = -\delta_{nn} d_n + b_{nk} + c_{nk}, \quad n, k \in \mathbb{N}_0,$$

where δ_{nk} denotes the Kronecker-delta symbol.

(E2) There exists $\alpha_* \in \mathbb{R}$ and, for all $\alpha > \alpha_*$, a constant $q(\alpha) \in (0, 1)$ with

$$e^{-\alpha k} \sum_{n=0}^{\infty} |b_{nk}| e^{\alpha n} \leq q(\alpha) d_k, \quad k \in \mathbb{N}_0, \quad \alpha > \alpha_*.$$

(E3) We have $\sup_{n \in \mathbb{N}} d_n e^{-\nu n} < \infty$ for all $\nu > 0$.

(E4) There exists a continuous increasing function $M : (\alpha_*, \infty) \rightarrow (0, \infty)$ such that

$$e^{-\alpha k} \sum_{n=0}^{\infty} |c_{nk}| e^{\alpha' n} \leq \frac{M(\alpha)}{\alpha - \alpha'}, \quad k \in \mathbb{N}_0, \quad \alpha' < \alpha.$$

Let us show that under conditions (E1) – (E4), equation (4.1) is a particular case of the results obtained in Section 3.

Theorem 4.2. *Suppose that (E1) – (E4) are satisfied. Then, for all $\alpha' < \alpha$ and $x \in \mathbb{E}_\alpha$, there exists a unique classical solution $(u(t))_{t \in [0, T(\alpha', \alpha))}$ in $\mathbb{E}_{\alpha'}$ with $T(\alpha', \alpha) = \frac{\alpha - \alpha'}{2eM(\alpha)}$ to (4.1).*

Proof. Define linear mappings A, B, C by

$$(Au)_n = -d_n u_n, \quad (Bu)_n = \sum_{k=0}^{\infty} b_{nk} u_k, \quad (Cu)_n = \sum_{k=0}^{\infty} c_{nk} u_k,$$

where $n \in \mathbb{N}_0$ and $u = (u_n)_{n=0}^\infty$ is such that the sums are absolutely convergent. In view of (E1), (4.1) is equivalent to

$$\frac{du_n(t)}{dt} = (Au(t))_n + (Bu(t))_n + (Cu(t))_n, \quad u_n(0) = x_n, \quad n \in \mathbb{N}_0.$$

Hence it suffices to show that Theorem 3.1 is applicable.

For $\alpha \in (\alpha_*, \alpha^*)$ let $(U_\alpha(t)u)_n := e^{-td_n}u_n$. Then $(U_\alpha(t))_{t \geq 0}$ is a holomorphic, positive semigroup with generator $(A_\alpha, D(A_\alpha))$ given by

$$(A_\alpha u)_n = (Au)_n = -d_n u_n, \quad u \in D(A_\alpha) = \{v \in \mathbb{E}_\alpha \mid (d_n u_n)_{n=0}^\infty \in \mathbb{E}_\alpha\}.$$

Define $(B_\alpha, D(A_\alpha))$ and $(B'_\alpha, D(A_\alpha))$ by

$$(B_\alpha u)_n = (Bu)_n = \sum_{k=0}^\infty b_{nk} u_k, \quad (B'_\alpha u)_n = \sum_{k=0}^\infty |b_{nk}| u_k, \quad u \in D(A_\alpha).$$

Then $(B'_\alpha, D(A_\alpha))$ is a well-defined positive operator satisfying

$$\|B'_\alpha u\|_\alpha \leq \sum_{k=0}^\infty e^{\alpha k} \left(\sum_{n=0}^\infty |b_{nk}| e^{\alpha n} \right) e^{-\alpha k} |u_k| \leq q(\alpha) \|A_\alpha u\|_\alpha, \quad u \in D(A_\alpha).$$

Hence by [33, Theorem 2.2] and [3, Theorem 1.1] it follows that $(A_\alpha + B'_\alpha, D(A_\alpha))$ is the generator of a holomorphic contraction semigroup on \mathbb{E}_α . Applying [3, Theorem 1.2] it follows that also $(A_\alpha + B_\alpha, D(A_\alpha))$ is the generator of a holomorphic contraction semigroup $(T_\alpha(t))_{t \geq 0}$ on \mathbb{E}_α . Then $(T_\alpha(t))_{t \geq 0}$ with generator $(A_\alpha + B_\alpha, D(A_\alpha))$ satisfies the conditions of Remark 2.5. Moreover, one has $A, B, C \in L(\mathbb{E})$ and C satisfies conditions (B1) and (B2). Hence Theorem 3.1 is applicable. \square

The next statement shows that the unique solution to (4.1) can be approximated by solutions u^N to certain finite-dimensional ordinary differential equations. Such result may be useful for numerical simulations.

Theorem 4.3. *Suppose that $(E1) - (E4)$ are satisfied. Define, for each $N \geq 1$, a new sequence $(a_{nk}^N)_{n,k=0}^\infty$ via $a_{nk}^N = -\delta_{nn}d_n^N + b_{nk}^N + c_{nk}^N$, by setting*

$$d_n^N = \mathbb{1}_{\{n \leq N\}} d_n, \quad b_{nk}^N = \mathbb{1}_{\{n,k \leq N\}} b_{nk}, \quad c_{nk}^N = \mathbb{1}_{\{n,k \leq N\}} c_{nk}.$$

Then, for all $\alpha' < \alpha$ and $x \in \mathbb{E}_\alpha$, there exists a unique classical solution $(u(t))_{0 \leq t < T(\alpha', \alpha)} \subset \mathbb{E}_{\alpha'}$ to (4.1) with $T(\alpha', \alpha) = \frac{\alpha - \alpha'}{2eM(\alpha)}$, and for all $N \geq 1$ there exist unique classical solutions $(u^N(t))_{0 \leq t < T(\alpha', \alpha)} \subset \mathbb{E}_{\alpha'}$ to

$$\frac{du_n^N(t)}{dt} = \sum_{k=0}^\infty a_{nk}^N u_k(t), \quad u_n^N(0) = x_n, \quad n \in \mathbb{N}_0.$$

Moreover it holds that

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T]} \sum_{n=0}^\infty |u_n^N(t) - u_n(t)| e^{\alpha n} = 0, \quad T \in (0, T(\alpha', \alpha)).$$

Proof. Apply Theorem 3.8. \square

Remark 4.4. Using the results of Section 3, we are also able to prove the existence of solutions to the adjoint equation

$$\frac{dv_n(t)}{dt} = \sum_{k=0}^{\infty} a_{kn} v_n(t), \quad v_n(0) = x_n, \quad n \in \mathbb{N}_0,$$

in the dual scale given by a weighted ℓ^∞ space. Uniqueness holds in such a case among all component-wise solutions.

It is not difficult to adapt such arguments to systems of Banach-space valued differential equations. Such equations arise naturally from the analysis of spatial birth-and-death processes, see, e.g., [5,25].

5. The spatial logistic model in the continuum

5.1. Description of the model

Let Γ be the space of all locally finite subsets of \mathbb{R}^d , see (1.1). We endow Γ with the smallest topology such that, for any continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ having compact support, $\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} f(x)$ is continuous. Then Γ is a Polish space, see, e.g., [27]. The space of finite configurations is defined by

$$\Gamma_0 = \bigsqcup_{n=0}^{\infty} \Gamma_0^{(n)}, \quad \Gamma_0^{(n)} = \{\eta \subset \mathbb{R}^d \mid |\eta| = n\}, \quad \Gamma_0^{(0)} = \{\emptyset\}.$$

We endow Γ_0 with the σ -algebra generated by cylinder sets of the form

$$\{\eta \in \Gamma_0 \mid |\eta \cap \Lambda| = n\}, \quad n \geq 0, \quad \Lambda \subset \mathbb{R}^d \text{ compact.}$$

Let $B_{bs}(\Gamma_0)$ be the space of all functions $G : \Gamma_0 \rightarrow \mathbb{R}$ such that

- (i) G is bounded and measurable.
- (ii) There exists $N(G) \in \mathbb{N}$ and a compact $\Lambda(G) \subset \mathbb{R}^d$ with $G(\eta) = 0$, whenever $|\eta| > N(G)$ or $\eta \cap \Lambda^c \neq \emptyset$.

The space of all polynomially bounded cylinder functions is defined by

$$\mathcal{FP}(\Gamma) = \left\{ F : \Gamma \rightarrow \mathbb{R} \mid \exists G \in B_{bs}(\Gamma_0) \text{ with } F(\gamma) = \sum_{\eta \in \gamma} G(\eta) \quad \forall \gamma \in \Gamma \right\},$$

where \in indicates that the sum runs only over all finite subsets of γ .

Remark 5.1. For each $F \in \mathcal{FP}(\Gamma)$ there exists $N(G) \in \mathbb{N}$, $A(G) \geq 0$ and a compact $\Lambda(G) \subset \mathbb{R}^d$ such that $F(\gamma) = F(\gamma \cap \Lambda)$ and $|F(\gamma)| \leq A(1 + |\gamma \cap \Lambda|)^N$.

Consider a (heuristic) Markov operator L acting on $\mathcal{FP}(\Gamma)$ via the formula

$$\begin{aligned} (LF)(\gamma) = & \sum_{x \in \gamma} \left(m + \sum_{y \in \gamma \setminus x} a^-(x-y) \right) (F(\gamma \setminus x) - F(\gamma)) \\ & + \sum_{x \in \gamma} \int_{\mathbb{R}^d} a^+(x-y) (F(\gamma^+ \cup y) - F(\gamma)) dy. \end{aligned}$$

For simplicity of notation, we have let $\gamma \setminus x$ and $\gamma \cup x$ stand for $\gamma \setminus \{x\}$ and $\gamma \cup \{x\}$. The first term describes the death of a particle located at $x \in \gamma$. Such a Markov event may be either caused by a state-independent mortality with parameter $m \geq 0$, or by an interaction with another particle at position $y \in \gamma \setminus x$ at rate $a^-(x-y) \geq 0$. The second term describes the branching mechanism where each particle at position $x \in \gamma$ may create a new particle at position $y \in \mathbb{R}^d$. The distribution and rate of such a Markov event is described by $a^+(x-y) \geq 0$.

Using Remark 5.1 it is easily seen that $LF(\gamma)$ is well-defined for any $\gamma \in \Gamma$ and $F \in \mathcal{FP}(\Gamma)$, provided that a^\pm are bounded and have compact support. Such model was studied in [12,25] where the following balance condition for the birth-and-death rates has been used

(G) $a^\pm \geq 0$ are symmetric, bounded, integrable and that there exists $\vartheta > 0$ and $b \geq 0$ such that

$$\sum_{x \in \eta} \sum_{y \in \eta \setminus x} (a^-(x-y) - \vartheta a^+(x-y)) \geq -b|\eta|, \quad \eta \in \Gamma_0. \quad (5.1)$$

Note that this condition does not imply that $LF(\gamma)$ is well-defined for all $\gamma \in \Gamma$. Hence we define LF in a different way explained below.

5.2. The Fokker-Planck equation

In the formulation of the corresponding Fokker-Planck equation (1.3) we will only require that LF is well-defined for μ -a.a $\gamma \in \Gamma$, where μ belongs to a certain class of states on Γ . Hence we do not assume that a^\pm have compact supports.

Definition 5.2. Let μ_0 be a Borel probability measure on Γ . A solution to (1.3) is a family of Borel probability measures $(\mu_t)_{t \geq 0}$ on Γ satisfying, for any $F \in \mathcal{FP}(\Gamma)$,

(a) $F, LF \in L^1(\Gamma, \mu_t)$ for all $t \geq 0$.

(b) $t \mapsto \int_{\Gamma} (LF)(\gamma) d\mu_t(\gamma)$ is locally integrable and

$$\int_{\Gamma} F(\gamma) d\mu_t(\gamma) = \int_{\Gamma} F(\gamma) d\mu_0(\gamma) + \int_0^t \int_{\Gamma} (LF)(\gamma) d\mu_s(\gamma) ds, \quad t \geq 0.$$

Below we study the Fokker-Planck equation in terms of the corresponding correlation function evolution obtained from (1.4). Namely, introduce the combinatorial transformation

$$(KG)(\gamma) := \sum_{\eta \in \gamma} G(\eta), \quad \gamma \in \Gamma, \quad G \in B_{bs}(\Gamma_0),$$

with inverse given by $(K^{-1}F)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi)$. For a Borel probability measure μ on Γ the correlation function $k_{\mu} : \Gamma_0 \rightarrow \mathbb{R}_+$ is uniquely determined by

$$\int_{\Gamma} (KG)(\gamma) d\mu(\gamma) = \int_{\Gamma_0} G(\eta) k_{\mu}(\eta) d\lambda(\eta), \quad G \in B_{bs}(\Gamma_0), \quad (5.2)$$

where λ denotes the Lebesgue-Poisson measure on Γ_0 defined by the relation

$$\int_{\Gamma_0} G(\eta) d\lambda(\eta) = G(\{\emptyset\}) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G(\{x_1, \dots, x_n\}) dx_1 \dots dx_n, \quad G \in B_{bs}(\Gamma_0).$$

Note that such correlation function does not need to exist. However, it is necessary and sufficient that μ has locally finite moments and is locally absolutely continuous with respect to the Poisson measure on Γ , see [26] and the references therein.

Let us explain how (1.4) can be derived from the Fokker-Planck equation, see [17] for additional details. Take μ such that it has correlation function k_{μ} and let $G \in B_{bs}(\Gamma_0)$. Then, at least formally, one obtains for $\hat{L} := K^{-1}LK$ the relation

$$\int_{\Gamma} (LKG)(\gamma) d\mu(\gamma) = \int_{\Gamma} (K\hat{L}G)(\gamma) d\mu(\gamma) = \int_{\Gamma_0} (\hat{L}G)(\eta) k_{\mu}(\eta) d\lambda(\eta). \quad (5.3)$$

A computation shows that \hat{L} is given by $\hat{L} = \hat{L}_0 + \hat{L}_1$ with

$$\begin{aligned} (\hat{L}_0 G)(\eta) &= - \left(m|\eta| + \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^-(x-y) \right) G(\eta) + \sum_{x \in \eta} \int_{\mathbb{R}^d} a^+(x-y) G(\eta \cup y) dy, \\ (\hat{L}_1 G)(\eta) &= - \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^-(x-y) G(\eta \setminus x) + \sum_{x \in \eta} \int_{\mathbb{R}^d} a^+(x-y) G(\eta \setminus x \cup y) dy. \end{aligned}$$

Hence using (5.2) and (5.3) we may reformulate (1.3) to

$$\frac{d}{dt} \int_{\Gamma_0} G(\eta) k_{\mu_t}(\eta) d\lambda(\eta) = \int_{\Gamma_0} (\widehat{L}G)(\eta) k_{\mu_t}(\eta) d\lambda(\eta), \quad G \in B_{bs}(\Gamma_0). \quad (5.4)$$

The operator L^Δ introduced in Section 1 should therefore be related with \widehat{L} by

$$\int_{\Gamma_0} (\widehat{L}G)(\eta) k(\eta) d\lambda(\eta) = \int_{\Gamma_0} G(\eta) (L^\Delta k)(\eta) d\lambda(\eta), \quad G, k \in B_{bs}(\Gamma_0). \quad (5.5)$$

It can be shown that it is given by $L^\Delta = L_0^\Delta + L_1^\Delta$, where

$$\begin{aligned} (L_0^\Delta k)(\eta) &= - \left(m|\eta| + \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^-(x-y) \right) k(\eta) + \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^+(x-y) k(\eta \setminus x) \\ (L_1^\Delta k)(\eta) &= - \sum_{x \in \eta_{\mathbb{R}^d}} \int a^-(x-y) k(\eta \cup y) dy + \sum_{x \in \eta_{\mathbb{R}^d}} \int a^+(x-y) k(\eta \setminus x \cup y) dy. \end{aligned}$$

Hence (5.4) is simply a weak formulation to (1.4).

5.3. Uniqueness for the Fokker-Planck equation

Below we introduce corresponding Banach spaces and study uniqueness for (1.3) and (5.4). Let \mathcal{L}_α be the Banach space of functions G equipped with the norm

$$\|G\|_{\mathcal{L}_\alpha} = \int_{\Gamma_0} |G(\eta)| e^{\alpha|\eta|} d\lambda(\eta).$$

Using the duality $\langle G, k \rangle = \int_{\Gamma_0} G(\eta) k(\eta) d\lambda(\eta)$ we may identify \mathcal{L}_α^* with the Banach space \mathcal{K}_α of functions k equipped with the norm

$$\|k\|_{\mathcal{K}_\alpha} = \operatorname{ess\,sup}_{\eta \in \Gamma_0} |k(\eta)| e^{-\alpha|\eta|}.$$

Observe that $\mathcal{L} = (\mathcal{L}_\alpha)_\alpha$ defines a scale of Banach spaces in the sense of (1.5) while $\mathcal{K} = (\mathcal{K}_\alpha)_\alpha$ corresponds to (3.14). Then we can show the following lemma.

Lemma 5.3. *Suppose that (G) is satisfied. Then*

(a) $\widehat{L}_0, \widehat{L}_1$ define operators in $L(\mathcal{L})$ such that, for all $\alpha' < \alpha$, one has

$$\|\widehat{L}_0\|_{L(\mathcal{L}_\alpha, \mathcal{L}_{\alpha'})} \leq \frac{m}{e(\alpha - \alpha')} + \frac{\|a^-\|_\infty + \|a^+\|_\infty}{4e^2(\alpha - \alpha')^2},$$

$$\|\widehat{L}_1 G\|_{L(\mathcal{L}_\alpha, \mathcal{L}_{\alpha'})} \leq \frac{\|a^-\|_{L^1} e^\alpha + \|a^+\|_{L^1}}{e(\alpha - \alpha')}.$$

(b) L_0^Δ, L_1^Δ define operators in $L(\mathcal{K})$ such that, for all $\alpha' < \alpha$, one has

$$\|\widehat{L}_0\|_{L(\mathcal{L}_\alpha, \mathcal{L}_{\alpha'})} = \|L_0^\Delta\|_{L(\mathcal{K}_\alpha, \mathcal{K}_\alpha)}, \quad \|\widehat{L}_1\|_{L(\mathcal{L}_\alpha, \mathcal{L}_{\alpha'})} = \|L_1^\Delta\|_{L(\mathcal{K}_\alpha, \mathcal{K}_\alpha)}.$$

Proof. Assertion (a) follows by a direct computation, see, e.g., [25] and the references therein. Assertion (b) is a consequence of the duality $\mathcal{L}_\alpha^* = \mathcal{K}_\alpha$. \square

The following is due to [25].

Theorem 5.4. Let $\alpha_0 \in \mathbb{R}$ and μ_0 be a probability measure on Γ having correlation function $k_0 \in \mathcal{K}_{\alpha_0}$. Then there exists a unique family $(k_t)_{t \geq 0} \subset \bigcup_{\alpha \in \mathbb{R}} \mathcal{K}_\alpha$ with the following properties:

(a) For each $T > 0$ there exists $\alpha_T \geq \alpha_0$ such that $k_t \in \mathcal{K}_{\alpha_T}$, $t \in [0, T]$ and $(k_t)_{t \in [0, T]}$ is the unique classical solution in \mathcal{K}_{α_T} to

$$\frac{\partial k_t}{\partial t} = L^\Delta k_t, \quad k_t|_{t=0} = k_0. \quad (5.6)$$

(b) There exists a unique family of Borel probability measures $(\mu_t)_{t \geq 0}$ on Γ such that, for any $t \geq 0$, k_t given by (a) is the correlation function of μ_t .

Note that existence and uniqueness is only established for classical solutions to (5.6). Although an evolution of states $(\mu_t)_{t \geq 0}$ was constructed, its relation to (1.3) was not considered there. Below we show that the results obtained in Section 3 can be applied to prove the following.

Theorem 5.5. Suppose that condition (G) is satisfied. Then

- (a) The family $(\mu_t)_{t \geq 0}$, constructed in Theorem 5.4, is a weak solution to (1.3).
- (b) Let $(\nu_t)_{t \geq 0}$ be another weak solution to (1.3) which admits a sequence of correlation functions $(k_{\nu_t})_{t \geq 0}$ and suppose that for any $T > 0$ there exists $\beta_T \geq \alpha_0$ with $\sup_{t \in [0, T]} \|k_{\nu_t}\|_{\mathcal{K}_{\beta_T}} < \infty$. Then $\mu_t = \nu_t$ for all $t \geq 0$.

Proof. (a) Let $(k_t)_{t \geq 0}$ be the family of correlation functions corresponding to $(\mu_t)_{t \geq 0}$ given by Theorem 5.4. Fix any $T > 0$ and let $\alpha_T \geq \alpha_0$ be such that $k_t \in \mathcal{K}_{\alpha_T}$. Take $F \in \mathcal{FP}(\Gamma)$ and let $G \in B_{bs}(\Gamma_0)$ be such that $F = KG$. Then

$$\int_{\Gamma} |F(\gamma)| d\mu_t(\gamma) \leq \int_{\Gamma} \sum_{\eta \in \gamma} |G(\eta)| d\mu(\gamma) = \int_{\Gamma_0} |G(\eta)| k_t(\eta) d\lambda(\eta) < \infty.$$

Since, for $\alpha'' < \alpha$, one has $B_{bs}(\Gamma_0) \subset \mathcal{L}_{\alpha''}$ we obtain $\widehat{L}G \in \mathcal{L}_\alpha$ and hence

$$\int_{\Gamma_0} |\widehat{L}G(\eta)| k_t(\eta) d\lambda(\eta) \leq \sup_{t \in [0, T]} \|k_t\|_{\mathcal{K}_{\alpha_T}} \int_{\Gamma_0} |\widehat{L}G(\eta)| e^{\alpha|\eta|} d\lambda(\eta) < \infty,$$

where we have used that $[0, T] \ni t \mapsto k_t \in \mathcal{K}_{\alpha_T}$ is continuously differentiable and hence bounded. This implies that $\widehat{L}G \in L^1(\Gamma_0, k_t \lambda)$. Since K can be uniquely extended to a bounded linear operator $K : L^1(\Gamma_0, k_t \lambda) \rightarrow L^1(\Gamma, \mu_t)$, see [26], it follows that $LF = K\widehat{L}G \in L^1(\Gamma, \mu_t)$. Finally by (5.2) we obtain

$$\int_{\Gamma} F(\gamma) d\mu_t(\gamma) = \int_{\Gamma_0} G(\eta) k_t(\eta) d\lambda(\eta), \quad (5.7)$$

and using the definition of \widehat{L} together with (5.5) yields

$$\int_{\Gamma} (LF)(\gamma) d\mu_t(\gamma) = \int_{\Gamma_0} (\widehat{L}G)(\eta) k_{\mu_t}(\eta) d\lambda(\eta) = \int_{\Gamma_0} G(\eta) (L^\Delta k_{\mu_t})(\eta) d\lambda(\eta). \quad (5.8)$$

Note that (5.5) was stated only for functions from $B_{bs}(\Gamma_0)$. However, by approximation it extends to all $k \in \mathcal{K}_{\alpha_T}$. In particular, it follows from Lemma 5.3(b) for $\beta > \alpha_T$

$$\begin{aligned} \left| \int_{\Gamma} LF(\gamma) d\mu_t(\gamma) \right| &\leq \int_{\Gamma_0} |G(\eta)| |L^\Delta k_t(\eta)| d\lambda(\eta) \\ &\leq \|L^\Delta\|_{L(\mathcal{K}_{\alpha_T}, \mathcal{K}_\beta)} \sup_{t \in [0, T]} \|k_t\|_{\mathcal{K}_{\alpha_T}} \int_{\Gamma_0} |G(\eta)| e^{\beta|\eta|} d\lambda(\eta) < \infty. \end{aligned}$$

Since $T > 0$ was arbitrary, we conclude that $(\mu_t)_t$ is a solution to (1.3).

(b) Conversely, let $(\nu_t)_{t \geq 0}$ be a solution to (1.3) with the desired properties. Using (5.7) and (5.8) in this particular case shows that k_{ν_t} satisfies (5.4). In particular, $t \mapsto \langle G, k_{\nu_t} \rangle$ is continuous for any $G \in B_{bs}(\Gamma_0)$. By approximation and since $\sup_{t \in [0, T]} \|k_{\nu_t}\|_{\mathcal{K}_{\beta_T}} < \infty$, we see that $t \mapsto \langle G, k_{\nu_t} \rangle$ is also continuous for any $G \in \mathcal{L}_{\beta_T}$. Thus it suffices to show that Theorem 3.17 is applicable. Indeed, write $\widehat{L} = \widehat{L}_{0,b} + \widehat{L}_{1,b}$, where

$$\widehat{L}_{0,b}G := \widehat{L}_0G - b|\eta|G, \quad \widehat{L}_{1,b}G := \widehat{L}_1G + b|\eta|G.$$

For $\beta > \ln(\vartheta)$ let $\mathcal{D}_\beta := \{G \in \mathcal{L}_\beta \mid M \cdot G \in \mathcal{L}_\beta\}$, where $M(\eta) = m|\eta| + \sum_{x \in \eta} \int_{y \in \eta \setminus x} a^-(x - y)$. Condition (5.1) implies, for any $0 \leq G \in \mathcal{D}_\beta$,

$$\int_{\Gamma_0} \sum_{x \in \eta} \int_{\mathbb{R}^d} |G(\eta \cup y)| a^+(x - y) dy e^{\beta|\eta|} d\lambda(\eta) \leq \frac{e^{-\beta}}{\vartheta} \int_{\Gamma_0} (M(\eta) + b|\eta|) |G(\eta)| e^{\beta|\eta|} d\lambda(\eta).$$

By [33] and [3] it follows that $(\widehat{L}_{0,b}, \mathcal{D}_\beta)$ is the generator of a positive, analytic semigroup $(S_\beta(t))_{t \geq 0}$ of contractions on \mathcal{L}_β . It is not difficult to see that $(S_\beta(t))_{t \geq 0}$ with generator satisfies Remark 2.5 for $\beta > \ln(\vartheta)$, see, e.g., [20] for a more general statement. Moreover, for any $\beta' < \beta$ and $G \in \mathcal{L}_\beta$, we obtain

$$\|\widehat{L}_{1,b}G\|_{\mathcal{L}_{\beta'}} \leq \frac{\langle a^- \rangle e^\alpha + b + \langle a^+ \rangle}{e(\alpha - \alpha')} \|G\|_{\mathcal{L}_\alpha}.$$

This shows that Theorem 3.1 is applicable in the time-homogeneous case with $\alpha_* = \ln(\vartheta)$, $\mathbb{E}_\alpha = \mathcal{L}_\alpha$, $A = L_{0,b}$ and $L_{1,b} = B$. Next, using Theorem 3.17 with $\mathcal{Y} = B_{bs}(\Gamma_0)$ gives the assertion. \square

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