Euler-Frobenius Numbers

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Abstract These numbers are defined as the coefficients of the Euler-Frobenius polynomials

$$P_{n,\lambda}(z) = \sum_{l=0}^{n} A_{n,l}(\lambda) z^l$$

which usually are introduced via the rational function expansion

$$\sum_{\nu=0}^{\infty} (\nu + \lambda)^n z^\nu = \frac{P_{n,\lambda}(z)}{(1-z)^{n+1}} \,,$$

n being a nonnegative integer and $\lambda \in [0, 1)$. The special case $A_{n,l}(0)$ is known from combinatorics (Eulerian numbers) and the general one $A_{n,l}(\lambda)$ occurs e.g. in approximation theory, summability, and rounding error analysis. Supplementing and extending known results on Eulerian numbers, various theorems for the Euler-Frobenius numbers $A_{n,l}(\lambda)$ and related quantities are established including unimodality, monotonicity properties and asymptotic expansions given by a local central limit theorem.

Keywords Eulerian numbers; Euler-Frobenius polynomials; Local central limit expansions; Rounding Errors

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1 Introduction and Summary

In this paper we are concerned with the coefficients of the Euler-Frobenius polynomials $P_{n,\lambda}$ which can be generated from the geometric series through the representations

$$\sum_{\nu=0}^{\infty} (\nu + \lambda)^n z^\nu = \left(\lambda + z \frac{d}{dz}\right)^n \frac{1}{1-z} = \frac{P_{n,\lambda}(z)}{(1-z)^{n+1}} \,, \tag{1.1}$$

n being a nonnegative integer and the parameter λ is considered to satisfy $\lambda \in [0,1)$, e.g. [27], p. 7, problem 46 in case $\lambda = 0$. For the power series, being convergent for $|z| < 1$, the two right hand expressions may serve as analytic extensions onto the punctured complex plane $\mathbb{C}\backslash\{1\}$. This function is a special case of Lerch's transcendental function, cf. [19] and [20], p. 33, which plays an important role in various parts of mathematics and related fields. For instance it occurs in the theory of analytic continuation of power series [15], summability [26], chapter IV, numerical analysis [28], structure of polymers [32], and in combinatorics [5], p. 51, and [9]. Some of these papers deal with asymptotics and the distribution of the zeros of $P_{n,\lambda}$ [7], [10], [11], [12], [16], [25], [26], [28], [30].

Starting from (1.1) straight forward computations lead to recursion formulae for the Euler-Frobenius polynomials $P_{n,\lambda}$ and its coefficients (see Lemmata 2.1, 2.2 below) giving

$$P_{n,\lambda}(z) = \sum_{l=0}^{n} A_{n,l}(\lambda) z^l \tag{1.2}$$

with $A_{n,l}(\lambda) \geq 0$ for $l = 0, \ldots, n$, $\lambda \in [0,1)$. In the sequel formally we put $A_{n,l}(\lambda) = 0$ for $l \notin \{0, \ldots, n\}$. For obvious reasons we call these coefficients **Euler-Frobenius numbers**. In the special case $\lambda = 0$ the numbers $A_{n,l}(0)$, $l = 1, \ldots, n$, are positive integers and they are termed Eulerian numbers in the literature. For instance they are well known from combinatorics where they count the permutations with precisely k rises in the symmetric group S_n [5], section 6.5, [9]. We emphasize that the Eulerian numbers are not to be confused with the Euler numbers E_n occurring in the power series expansion

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n, \quad |z| < \frac{\pi}{2},$$

for the reciprocal of the hyperbolic cosine.

At present the literature concerning analytic properties of $A_{n,l}(\lambda)$ primarily deals with Eulerian numbers $A_{n,l}(0)$ only where essentially inequalities and asymptotic formulae are derived [1], [3], [4], [5], [18], [29], [31]. It is the main purpose of this paper to generalize and to sharpen some of these results for the Euler-Frobenius numbers. Extending a well-known result for $A_{n,l}(0)$, e.g. [18] or [5], p. 292, problem 3, we prove that the finite sequence

$$A_{n,0}(\lambda), A_{n,1}(\lambda), \ldots, A_{n,n}(\lambda), \quad \lambda \in [0,1),$$

is unimodal (see Theorem 2.3 below). Our main result, Theorem 4.3, gives an asymptotic expansion of the type ($k \geq 3$ being an arbitrary integer)

$$\sqrt{\frac{n+1}{12}}\, \frac{A_{n,l}(\lambda)}{n!} = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \left(1 + \sum_{\mu=1}^{[(k-2)/2]} \frac{p_{4\mu}(x)}{(n+1)^{\mu}} \right) + o\left(\frac{1}{n^{(k-2)/2}} \right), \tag{1.3}$$

as $n \to \infty$, with explicitly computable even polynomials $p_{4\mu}$ of the quantity

$$x = \left(l + \lambda - \frac{n+1}{2} \right) \sqrt{\frac{12}{n+1}}$$

the degrees of which are at most 4μ. For a discussion of expansions of the kind (1.3) see the remarks following Lemma 3.1 below. Here and throughout $[\xi]$ denotes the largest integer not exceeding the real number ξ. In particular the remainder term in (1.3) holds uniformly with respect to $l \in \mathbb{Z}$. Thereby a result of Siraždinov for Eulerian numbers $A_{n,l}(0)$ [31] is extended to an asymptotic expansion and to the Euler-Frobenius numbers $A_{n,l}(\lambda)$ as well. In establishing (1.3) the basic tools are taken from the central limit theory of probability and from special functions by the so called Lindelöf-Wirtinger expansion of the particular case (1.1) of Lerch's transcendental function (Lemma 2.4). This approach to asymptotics occasionally is applied systematically to various special functions in the literature, e.g. [33], chapter 3.

Finally in section 5 we apply our main result (1.3) to the study of the probabilities

$$R_n = \frac{1}{(n-1)!} \sum_{j=0}^{[n/2]} (-1)^j \binom{n}{j} \left(\frac{n}{2} - j\right)^{n-1}, \; n \in \mathbb{N},$$

for a standard rounding problem occurring e.g. in the mathematics of elections [17], p. 185.

2 Elementary properties and analytic tools

In this section we collect some analytic facts being relevant in the sequel. Either the results are known or they can be derived in a straight forward manner. Therefore, in most cases we omit a detailed proof. In (1.1) the existence of the Euler-Frobenius polynomials $P_{n,\lambda}$ is readily verified by induction. At a first stage it follows that the degree of $P_{n,\lambda}$ is at most n. Some of the subsequent formulae hold for all $\lambda \in \mathbb{C}$, however, some properties require the assumption $\lambda \in [0, 1)$. Thus we assume the latter condition throughout the paper.

Lemma 2.1. *If $n \in \mathbb{N}_0$ and $\lambda \in [0, 1)$, then*

i) for $z \in \mathbb{C}$ we have
$P_{n+1,\lambda}(z) = \big(\lambda(1-z) + (n+1)z\big)P_{n,\lambda}(z) + z(1-z)P'_{n,\lambda}(z), \quad P_{0,\lambda}(z) = 1,$

ii) $P_{n,\lambda}(1) = n!$,

iii) all zeros of $P_{n,\lambda}$ are real, simple and nonpositive.

We only mention that a proof of part iii) is contained in [16], [25]. Moreover, the assertion iii) no longer holds, if the assumption $\lambda \in [0, 1)$ is dropped. On the basis of the recurrence relation in Lemma 2.1, i) and (1.2) we obtain the following properties of the Euler-Frobenius numbers in

Lemma 2.2. *If $n \in \mathbb{N}_0$ and $\lambda \in [0, 1)$, then we have*

i)

$$\begin{aligned} &A_{n+1,0}(\lambda) = \lambda A_{n,0}(\lambda) \;, \quad A_{0,0}(\lambda) = 1, \\ &A_{n+1,l}(\lambda) = (\lambda + l)A_{n,l}(\lambda) + (n + 2 - l - \lambda)A_{n,l-1}(\lambda), \quad 1 \le l \le n, \\ &A_{n+1,n+1}(\lambda) = (1-\lambda)A_{n,n}(\lambda), \; A_{0,0}(\lambda) = 1, \end{aligned}$$

in particular $A_{n,0}(\lambda) = \lambda^n$ and $A_{n,n}(\lambda) = (1-\lambda)^n$, *ii)* $A_{n,l}(\lambda) \geq 0$, $0 \leq l \leq n$,

iii) $A_{n,l}(\lambda) = \sum\limits_{j=0}^{l} (-1)^j \binom{n+1}{j} (l+\lambda-j)^n, \ l \geq 0,$

iv) $A_{n,l}(\lambda) = A_{n,n-l}(1-\lambda), \ 0 \leq l \leq n,$

v) $(z+1-\lambda)^n = \sum\limits_{l=0}^{n} A_{n,l}(\lambda) \binom{z+l}{n}, \ z \in \mathbb{C}.$

All parts of Lemma 2.2 generalize known formulae for the Eulerian case and v) in particular extends Worpitzky's identity [5], section 6.5. Further, from part i) we infer that n is the precise degree of $P_{n,\lambda}$, since $\lambda \neq 1$. Combining (1.2) and Lemma 2.1, ii) for later reference we record the property

$$\sum_{l=0}^{n} A_{n,l}(\lambda) = n!, \ n \in \mathbb{N}_0, \ \lambda \in [0,1). \tag{2.1}$$

Next we assume that the reader is familiar with the concept of unimodality for sequences, e.g. [5], section 7.1. Using Lemma 2.1, iii) in combination with a well known criterion for unimodality, e.g. [5], Theorem B, p. 270, we immediately obtain an extension of the unimodality property of the Eulerian numbers [5], p. 292, problem 3, [18] in

Theorem 2.3. *If $n \geq 3, \ \lambda \in [0,1)$, then the Euler-Frobenius numbers $A_{n,l}(\lambda)$ are unimodal with either a peak or a plateau of two points.*

Theorem 2.3 means that there is a number $l_0 \in \{1, \ldots, n-1\}$ such that

$$A_{n,0}(\lambda) \leq \ldots \leq A_{n,l_0-1}(\lambda) < A_{n,l_0}(\lambda) > A_{n,l_0+1}(\lambda) \geq \ldots \geq A_{n,n}(\lambda)$$

or

$$A_{n,0}(\lambda) \leq \ldots \leq A_{n,l_0-1}(\lambda) < A_{n,l_0}(\lambda) = A_{n,l_0+1}(\lambda) > A_{n,l_0+2}(\lambda) \geq \ldots \geq A_{n,n}(\lambda).$$

Next we give a representation of the rational function in (1.1) by means of the Lindelöf-Wirtinger expansion which turns out to be a useful tool for deriving our main result in section 4 [19], [20], p. 34, [34].

Lemma 2.4. *If $n \in \mathbb{N}_0$ and $\lambda \in [0,1)$, then for $z \in \mathbb{C} \backslash \{1\}$ we have*

$$\frac{P_{n,\lambda}(z)}{(1-z)^{n+1}} = n! \sum_{m=-\infty}^{\infty} \frac{e^{\lambda\left(2\pi i m + \log(1/z)\right)}}{\left(2\pi i m + \log(1/z)\right)^{n+1}} \ . \tag{2.2}$$

In (2.2) for $\log(1/z)$ we may choose that branch with $\operatorname{Im} \log(1/z) \in [0, 2\pi)$. Then the analyticity on the punctured cut $(0,\infty)\backslash\{1\}$ is generated by the summation over all branches of the logarithm. At $z=0$ the right hand side is defined by continuity.

3 Probabilistic tools from central limit theory

From Lemma 2.2, iii) for fixed $l \geq 0$ such that $l + \lambda > 0$ immediately we infer the trivial statement

$$A_{n,l}(\lambda) \sim (l+\lambda)^n, \text{ as } n \to \infty,$$

meaning that the ratio of both sides tends to 1. More interesting are asymptotic formulae for $A_{n,l}(\lambda)$, if l varies with n suitably. Actually such properties are known for special subsequences $l = l(n)$ and if $\lambda = 0$, e.g. [1], [4], [29]. Moreover, Siraždinov [31] obtained an asymptotic form for $A_{n,l}(0)$, as $n \to \infty$, holding uniformly with respect to $l \in \{0, \ldots, n\}$ (see also section 4 below). In order to extend his result to the case $\lambda \in [0,1)$ and to achieve an asymptotic expansion for $A_{n,l}(\lambda)$ in the sequel we follow the probabilistic approach of the above mentioned papers.
To fix ideas, from Lemmata 2.1, iii) and 2.2, i) we infer the representation

$$P_{n,\lambda}(z) = \sum_{l=0}^{n} A_{n,l}(\lambda) z^l = (1-\lambda)^n \prod_{\nu=1}^{n} \big(z + x_{n\nu}(\lambda)\big), \ \lambda \in [0,1) \tag{3.1}$$

with

$$0 \leq x_{n1}(\lambda) < x_{n2}(\lambda) < \ldots < x_{nn}(\lambda).$$

Using Lemma 2.1, ii) this may be rewritten as

$$\frac{P_{n,\lambda}(z)}{P_{n,\lambda}(1)} = \frac{P_{n,\lambda}(z)}{n!} = \prod_{\nu=1}^{n} \big(p_{n\nu}(\lambda) z + \tilde{p}_{n\nu}(\lambda)\big) \tag{3.2}$$

where

$$p_{n\nu}(\lambda) := \frac{1}{1 + x_{n\nu}(\lambda)}, \ \tilde{p}_{n\nu}(\lambda) := \frac{x_{n\nu}(\lambda)}{1 + x_{n\nu}(\lambda)} \ . \tag{3.3}$$

Thus the polynomials in (3.2) may be regarded as the generating function of the row sums of a triangular array of Bernoulli random variables

$$\begin{array}{lllll} X_{11} & & & & \\ X_{21} & X_{22} & & & \\ \vdots & & \ddots & & \\ X_{n1} & X_{n2} & \ldots & X_{nn} & \\ \vdots & \vdots & & & \ddots \end{array} \tag{3.4}$$

Here the entries are row-wise independent with distributions given by

$$P(X_{n\nu} = 1) = \pi_{n\nu}, \quad P(X_{n\nu} = 0) = 1 - \pi_{n\nu}, \tag{3.5}$$

where $\pi_{n\nu} \in [0,1]$ are given numbers.

In this section we derive an asymptotic expansion in a local central limit theorem for a general triangular Bernoulli array of the type given in (3.4), (3.5) that is an asymptotic expansion for

$$p(n,l) := P(S_n = l) \tag{3.6}$$

holding uniformly with respect to $l \in \mathbb{Z}$, where

$$S_n := \sum_{\nu=1}^{n} X_{n\nu} \tag{3.7}$$

denotes the row sums in the scheme (3.4). As an application in section 4 we consider the special case $\pi_{n\nu} = p_{n\nu}$ given by (3.3) for which it is possible to work out details by computing explicitly the coefficients of the asymptotic expansion.

To a large extent we follow the standard monograph of Petrov [24], chapter VII. There among others local central limit theorems for "simple sums" of the type $\sum_{\nu=1}^{n} X_\nu$ are proved, where $(X_\nu)_1^\infty$ is a sequence of independent random variables each X_ν having a lattice distribution and satisfying a moment condition. Here we consider the row sums (3.7) for the Bernoulli array (3.4). Although this case is not covered by Petrov in [24], chapters VI , VII, his proofs, however, can be modified by a few changes only in order to get asymptotic expansions for the probabilities (3.6). The basic case of a local limit theorem for "simple sums" $\sum_{\nu=1}^{n} X_\nu$ where each component X_ν has a lattice distribution is contained in Feller's standard monograph [8].

To start with we introduce some further pertinent notations and conditions for the Bernoulli scheme (3.4) with (3.5) - (3.7). We have for the expectation

$$E(S_n) = \sum_{\nu=1}^{n} E(X_{n\nu}) = \sum_{\nu=1}^{n} \pi_{n\nu}$$

and the variance

$$B_n := Var(S_n) = E(S_n^2) - E(S_n)^2$$

for which we assume that there exists an $n_0 \in \mathbb{N}$ such that

$$B_n \geq g\,n, \text{ for all } n \geq n_0, \tag{3.8}$$

g being a positive constant. Further we define the normalized cumulant of S_n by

$$\lambda_{\nu,n} := \frac{n^{(\nu-2)/2}}{B_n^{\nu/2}} \frac{1}{i^\nu} \left(\frac{d}{dt}\right)^\nu \log E(e^{itS_n})\Bigg|_{t=0} \tag{3.9}$$

$n, \nu \in \mathbb{N},\ \nu \geq 2$, where $E(e^{itS_n})$ is the characteristic function of S_n and log is that branch of the logarithm on the cut plane $\mathbb{C}\backslash(-\infty, 0]$ satisfying $\log 1 = 0$. Finally we introduce the functions

$$q_{\nu,n}(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{\substack{\mu_1,\ldots,\mu_\nu \geq 0 \\ \mu_1+2\mu_2+\ldots+\nu\mu_\nu=\nu}} H_{\nu+2s}(x) \prod_{m=1}^{\nu} \frac{1}{\mu_m!}\left(\frac{\lambda_{m+2,n}}{(m+2)!}\right)^{\mu_m}, \tag{3.10}$$

$x \in \mathbb{R},\ n, \nu \in \mathbb{N}$, where $s = \mu_1 + \ldots + \mu_\nu$. Further here the Hermite polynomials

$$H_m(x) := (-1)^m\ e^{x^2/2} \left(\frac{d}{dx}\right)^m\ e^{-x^2/2}, \tag{3.11}$$

for $x \in \mathbb{R},\ m \in \mathbb{N}_0$, are given by the Rodrigues formula using the scaling as customary in probability theory [24], p. 137. Now we can state the following asymptotic expansion for the quantities $p(n,l)$ in (3.6) as a local central limit theorem for triangular Bernoulli arrays.

Lemma 3.1. *Assuming the above notations and the condition (3.8), then for every $k \geq 3$ we have*

$$\sqrt{B_n}\ p(n,l) = \frac{1}{\sqrt{2\pi}}\ e^{-x^2/2} + \sum_{\nu=1}^{k-2} \frac{q_{\nu,n}(x)}{n^{\nu/2}} + o\left(\frac{1}{n^{(k-2)/2}}\right), \tag{3.12}$$

as $n \to \infty$, uniformly in $l \in \mathbb{Z}$, where $x = \big(l - E(S_n)\big)/\sqrt{B_n}$.

Although (3.12) holds uniformly with respect to $l \in \mathbb{Z}$ we emphasize the most valuable information is provided for l such that the quantity x remains bounded. This implies that the o-term is an error term indeed. If l is such that x becomes "large" then (3.12) may only give the estimate $p(n,l) = o\left(n^{-(k-2)/2}\right)$.

Proof. The random variables $X_{n\nu}$ in (3.4) obviously possess moments of all orders and thus the cumulants $\lambda_{\nu,n}$ in (3.9) exist for every $n, \nu \in \mathbb{N}$. As already mentioned above essentially the proof is contained in [24], chapters VI, VII. More precisely we have to adjust the proof of Theorem 12 in chapter VII, §3, pp. 204 /205 of [24] to the triangular array (3.4) with (3.5). To do so it turns out that basically the error estimates in Lemmata 11 and 12 in chapter VI of [24] have to be changed slightly. This requires some intrinsic analysis, however, the necessary estimations are straightforward. Therefore, the details are omitted here, cf. [22]. □

4 Asymptotics for the Euler-Frobenius numbers

In this section we derive the main result of our paper. This will be accomplished by an application of Lemma 3.1 to the special case $\pi_{n\nu} = p_{n\nu}(\lambda)$ given by (3.3). Observing (3.1) - (3.7) we have

$$p(n,l) = P(S_n = l) = \frac{A_{n,l}(\lambda)}{n!},\ l = 0, \ldots, n. \tag{4.1}$$

For this distribution of the row sums we compute moments and cumulants in order to make the resulting expansions as explicit as possible.

Lemma 4.1. *If $\lambda \in [0,1)$ and $n \geq 2$, then we have*

$$E(S_n) = \frac{n+1}{2} - \lambda \quad \textit{and} \quad B_n = Var(S_n) = \frac{n+1}{12}. \tag{4.2}$$

Proof. By means of Lemma 2.4 the characteristic function of the distribution given in (4.1) can be written as

$$E(e^{itS_n}) = \frac{P_{n,\lambda}(e^{it})}{n!} = \sum_{m=-\infty}^{\infty} e^{(2\pi m - t)i\lambda}\, T_m(t)^{n+1},\ t \in \mathbb{R}, \tag{4.3}$$

where

$$T_m(t) := \frac{1-e^{it}}{2\pi i m - it}\ ,\ m \in \mathbb{Z}.$$

Then T_m is an entire function for all $m \in \mathbb{Z}$ with $T_m(0) = 0$, if $m \neq 0$, and

$$T_0^{(\nu)}(0) = \frac{i^\nu}{\nu+1}\ ,\quad \nu \in \mathbb{N}_0.$$

Now by well known identities from probability theory we obtain

$$\begin{aligned} E(S_n) &= \frac{1}{i}\frac{d}{dt}\ E(e^{itS_n})\Big|_{t=0} \\ &= \frac{1}{i}\sum_{m=-\infty}^{\infty} e^{(2\pi m - t)i\lambda}\Big(-i\lambda T_m(t)^{n+1} + (n+1)T_m(t)^n\ T_m'(t)\Big)\Big|_{t=0} \\ &= \frac{n+1}{2} - \lambda \end{aligned}$$

and further (observe that $n \geq 2$)

$$\begin{aligned} B_n = Var(S_n) &= \frac{1}{i^2}\left(\frac{d}{dt}\right)^2\ E\big(e^{it(S_n - E(S_n))}\big)\Big|_{t=0} \\ &= -\left(\frac{d}{dt}\right)^2\ E\big(e^{it(S_n+\lambda-(n+1)/2)}\big)\Big|_{t=0} \\ &= -\left(\frac{d}{dt}\right)^2\ \sum_{m=-\infty}^{\infty} e^{2\pi i m\lambda}\ e^{-it(n+1)/2}\ T_m(t)^{n+1}\Big|_{t=0} \\ &= -\sum_{m=-\infty}^{\infty} e^{2\pi i m\lambda}\ e^{-it(n+1)/2)}\Bigg(-\left(\frac{n+1}{2}\right)^2 T_m(t)^{n+1} + \\ &\quad +2\Big(-i\frac{n+1}{2}\Big)(n+1)T_m(t)^n T_m'(t) + (n+1)nT_m(t)^{n-1}T_m'(t)^2 + (n+1)T_m(t)^n T_m''(t)\Bigg)\Bigg|_{t=0} \\ &= \frac{n+1}{12}\ . \end{aligned}$$

□

Lemma 4.2. *Suppose that $\lambda \in [0,1), n, \mu \in \mathbb{N}$, and B_n is given by (4.2). Then we have* *i)*

$$\begin{aligned} \lambda_{2\mu+1,n} &= 0, & n \geq 2\mu+1 \geq 3, \\ \lambda_{2\mu,n} &= \left(\frac{n}{B_n}\right)^{\mu-1} \frac{6}{\mu}\, \mathcal{B}_{2\mu}, & n \geq 2\mu \geq 2, \end{aligned} \tag{4.4}$$

where $\mathcal{B}_\nu$, $\nu \in \mathbb{N}_0$, denote the Bernoulli numbers, e.g. [13], p.22, in particular $\lambda_{2,n} = 1$ [22], p. 134.

ii)

$$q_{2\mu-1,n}(x) = 0, \quad n \geq 2\mu+1 \geq 3,$$

$$q_{2\mu,n}(x) = \left(\frac{n}{B_n}\right)^{\mu} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \sum_{k_2+2k_4+\ldots+\mu k_{2\mu}=\mu} H_{2(\mu+s)}(x) 6^s \prod_{r=1}^{\mu} \frac{1}{k_{2r}!} \left(\frac{\mathcal{B}_{2(r+1)}}{(2r+2)!(r+1)}\right)^{k_{2r}}, \tag{4.5}$$

$n \geq 2\mu+2 \geq 4$, where $s = k_2 + k_4 + \ldots + k_{2\mu}$, and H_m being the Hermite polynomials defined in (3.11).

Proof. i) We start from the definition of $\lambda_{\nu,n}$ given by (3.9) and use the characteristic function for the distribution in (4.1) given by the representation (4.3). Observing (4.2) we get

$$\begin{aligned} E\left(e^{it\left(S_n+\lambda-(n+1)/2\right)}\right) &= \sum_{m=-\infty}^{\infty} e^{-it(n+1)/2}\, T_m(t)^{n+1} \\ &= \sum_{m=-\infty}^{\infty} e^{2\pi i m\lambda} \left(\frac{e^{it/2}-e^{it/2}}{it-2\pi i m}\right)^{n+1} \\ &= \left(\frac{\sin(t/2)}{t/2}\right)^{n+1} \left(1 + \sum_{m\neq 0} e^{2\pi i m\lambda} \Big(\frac{t}{t-2\pi m}\Big)^{n+1}\right) \\ &=: \left(\frac{\sin(t/2)}{t/2}\right)^{n+1} \left(1 + t^{n+1} h_n(t)\right), \end{aligned}$$

the functions h_n being holomorphic for $|t| < 2\pi$. Thus for $n \geq \nu \geq 2$ we get (observe (4.2))

$$\begin{aligned} \lambda_{\nu,n} &= \frac{n^{(\nu-2)/2}}{B_n^{\nu/2}} \frac{1}{i^\nu} \left(\frac{d}{dt}\right)^{\nu} \log E\left(e^{it\left(S_n+\lambda-(n+1)/2\right)}\right)\Bigg|_{t=0} \\ &= \frac{n^{(\nu-2)/2}}{B_n^{\nu/2}} \frac{n+1}{i^\nu} \left(\frac{d}{dt}\right)^{\nu} \log \frac{\sin(t/2)}{t/2}\Bigg|_{t=0} \\ &= \left(\frac{n}{B_n}\right)^{(\nu-2)/2} \frac{12}{i^\nu} \left(\frac{d}{dt}\right)^{\nu-1} \frac{1}{2}\left(\operatorname{cotan}\frac{t}{2} - \frac{2}{t}\right)\Bigg|_{t=0}. \end{aligned}$$

Next using the expansion, see [13], p.35,

$$\operatorname{cotan} z - \frac{1}{z} = \sum_{\mu=1}^{\infty} (-1)^\mu \frac{4^\mu}{(2\mu)!} \mathcal{B}_{2\mu} z^{2\mu-1} \quad , \quad 0 < |z| < \pi,$$

we may proceed by

$$\lambda_{\nu,n} = \left(\frac{n}{B_n}\right)^{(\nu-2)/2} \frac{12}{i^\nu} \sum_{\mu=1}^{\infty} (-1)^\mu \frac{\mathcal{B}_{2\mu}}{(2\mu)!} (2\mu-1)\cdots(2\mu-\nu+1) t^{2\mu-\nu} \Bigg|_{t=0}$$

$$= \begin{cases} 0 & , \quad n \geq \nu = 2\mu+1 \geq 3 \\ \left(\frac{n}{B_n}\right)^{\mu-1} \frac{6}{\mu} \mathcal{B}_{2\mu} & , \quad n \geq \nu = 2\mu \geq 2 \; . \end{cases}$$

ii) Suppose that $\nu = 2\mu - 1$ is odd and $n \geq 2\mu + 1 \geq 3$, then by (3.10) we have

$$q_{2\mu-1,n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{k_1+2k_2+\ldots+(2\mu-1)k_{2\mu-1}=2\mu-1} H_{2\mu-1+2s}(x) \prod_{m=1}^{2\mu-1} \frac{1}{k_m!} \left(\frac{\lambda_{m+2,n}}{(m+2)!}\right)^{k_m},$$

where $s = k_1 + k_2 + \ldots + k_{2\mu-1}$. For every solution $(k_1, \ldots, k_{2\mu-1})$ of the equation $k_1 + 2k_2 + \ldots + (2\mu-1)k_{2\mu-1} = 2\mu - 1$ obviously there is an odd index $m_0 \in \{1, \ldots, 2\mu-1\}$ such that $k_{m_0} > 0$. Hence, by (4.4), we have $\lambda_{m_0+2,n} = 0$ and thus $q_{2\mu-1,n}(x)$ vanishes identically. Next suppose that $\nu = 2\mu$ is even and $n \geq 2\mu + 2$. Again by (3.10) we get $(s = k_1 + k_2 + \ldots + k_{2\mu})$

$$q_{2\mu,n}(x) == \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{k_1+2k_2+\ldots+2\mu k_{2\mu}=2\mu} H_{2(\mu+s)}(x) \prod_{m=1}^{2\mu} \frac{1}{k_m!} \left(\frac{\lambda_{m+2,n}}{(m+2)!}\right)^{k_m}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{k_2+2k_4+\ldots+\mu k_{2\mu}=\mu} H_{2(\mu+s)}(x) \prod_{r=1}^{\mu} \frac{1}{k_{2r}!} \left(\frac{\lambda_{2(r+1),n}}{(2r+2)!}\right)^{k_{2r}},$$

where $s = k_2 + k_4 + \ldots + k_{2\mu}$. The last identity follows from (4.4) and an analogous argument as in the odd case above. Finally, using (4.4) again, we conclude

$$\prod_{r=1}^{\mu} \left(\lambda_{2(r+1),n}\right)^{k_{2r}} = \prod_{r=1}^{\mu} \left(\left(\frac{n}{B_n}\right)^r \frac{6}{r+1} \mathcal{B}_{2(r+1)}\right)^{k_{2r}}$$

$$= \left(\frac{n}{B_n}\right)^\mu 6^s \prod_{r=1}^{\mu} \left(\frac{\mathcal{B}_{2(r+1)}}{r+1}\right)^{k_{2r}}$$

and the representation for $q_{2\mu,n}(x)$ follows.

□

Now we are prepared to state our main result by the following expansion for the Euler-Frobenius numbers in the sense of a local central limit theorem.

Theorem 4.3. *Suppose that $\lambda \in [0, 1)$ and B_n is given by (4.2). Then for every $k \geq 3$ the Euler-Frobenius numbers $A_{n,l}(\lambda)$ satisfy*

$$\frac{\sqrt{B_n}}{n!} A_{n,l}(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \sum_{\mu=1}^{[(k-2)/2]} \frac{\tilde{q}_{2\mu}(x)}{B_n^\mu} + o\left(\frac{1}{n^{(k-2)/2}}\right) \tag{4.6}$$

as $n \to \infty$, uniformly in $l \in \mathbb{Z}$ with

$$x = \left(l + \lambda - \frac{n+1}{2}\right) \sqrt{\frac{12}{n+1}} \tag{4.7}$$

and

$$\tilde{q}_{2\mu}(x) := \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \sum_{k_2+2k_4+\ldots+\mu k_{2\mu}=\mu} H_{2(\mu+s)}(x)\; 6^s \prod_{r=1}^{\mu} \frac{1}{k_{2r}!} \left(\frac{\mathcal{B}_{2(r+1)}}{(2r+2)!(r+1)}\right)^{k_{2r}} \tag{4.8}$$

where $s = k_2 + k_4 + \ldots + k_{2\mu}$ and $\mathcal{B}_\nu$ denote the Bernoulli numbers and H_m are the Hermite polynomials defined in (3.11).

Proof. We may apply Lemma 3.1 to the distribution in (4.1), since, by (4.2), the condition (3.8) is satisfied with $g = 1/12$ and $n_0 = 1$. In particular we obtain (4.6) from Lemmata 4.1 and 4.2, ii). □

We exhibit the case $k = 7$ of (4.6) explicitly in

Corollary 4.4. *If $\lambda \in [0,1)$ and x is given in (4.7) then we have*

$$\sqrt{\frac{n+1}{12}}\frac{1}{n!}\; A_{n,l}(\lambda) = \frac{1}{\sqrt{2\pi}}\; e^{-\frac{x^2}{2}} \left(1 - \frac{x^4-6x^2+3}{20(n+1)} + \frac{x^6-15x^4+45x^2-15}{105(n+1)^2} + \frac{x^8-28x^6+210x^4-420x^2+105}{800(n+1)^2}\right) + o\left(\frac{1}{n^{5/2}}\right),$$

as $n \to \infty$, uniformly in $l \in \mathbb{Z}$.

We mention that Theorem 4.3 gives a considerable extension of a result of Siraždinov [31] who proved an asymptotic form for $A_{n,l}(0)$ which essentially corresponds to the special case $\lambda = 0$ and $k = 4$ of Theorem 4.3.

Finally we conclude this section by the statement that the Euler-Frobenius numbers are asymptotically normal, thereby supplementing the known case $\lambda = 0$ (see [1], [4]). Since a proof uses a routine argument on the basis of Lyapunov's theorem [29], p.23, and Lemma 4.1 we omit a detailed explanation.

Theorem 4.5. *If $\lambda \in [0,1)$, then for all $x \in \mathbb{R}$ we have*

$$\lim_{n\to\infty} \frac{1}{n!} \sum_{l<\sqrt{\frac{n+1}{12}}x+\frac{n+1}{2}-\lambda} A_{n,l}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

5 Rounding errors

This final section is devoted to the analysis of a sequence of numbers which is closely related to the Euler-Frobenius numbers with parameters $\lambda = 0, \frac{1}{2}$. To begin with we consider the following problem for rounding probabilities. Suppose that $a_1, \ldots, a_n$ are positive numbers with sum being an integer. For the numbers $\tilde{a}_1, \ldots, \tilde{a}_n$ generated by standard rounding according to

$$\tilde{a}_j := \begin{cases} [a_j] & \text{if } a_j < [a_j] + \frac{1}{2} \\ [a_j] + 1 & \text{if } a_j \geq [a_j] + \frac{1}{2}, \end{cases}$$

$j = 1, \ldots, n$, let R_n be the probability that $\sum\limits_{j=1}^{n} a_j = \sum\limits_{j=1}^{n} \tilde{a}_j$. It is known by comparing volumes of simplices [17], p.185, that

$$R_n = \frac{1}{(n-1)!} \sum_{j=0}^{[n/2]} (-1)^j \binom{n}{j} \left(\frac{n}{2} - j\right)^{n-1}, \; n \geq 1. \tag{5.1}$$

In the sequel we study monotonicity properties and asymptotics for the sequence (R_n). First we identify these numbers with Euler-Frobenius numbers and we derive an integral representation in

Lemma 5.1. *We have*

i)
$$R_{2k+1} = \frac{1}{(2k)!} \; A_{2k,k}\left(\frac{1}{2}\right), \; R_{2k} = \frac{1}{(2k-1)!} \; A_{2k-1,k} \; (0), \; k \in \mathbb{N},$$

ii)
$$R_n = \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin t}{t}\right)^n dt \; , \; n \in \mathbb{N}.$$

Proof. The first part immediately follows from (5.1) and Lemma 2.2, iii). For ii) we use the well-known relationship between the probabilities of a lattice distribution and its characteristic function, e.g. [24], Theorem 6, p.12. Using (4.1) and (4.3) this gives

$$\begin{aligned} \frac{A_{n,l}(\lambda)}{n!} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} e^{(2\pi m - t)i\lambda} \; T_m(t)^{n+1} e^{-ilt} dt \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{(2\pi m - t)i\lambda} \left(\frac{1 - e^{it}}{2\pi i m - it}\right)^{n+1} e^{-ilt} \; dt, \end{aligned} \tag{5.2}$$

for $0 \leq l \leq n$. Obviously, by (5.1), the representation for R_1 is correct. Next let $n = 2k$ be even, then from i) and (5.2) we conclude

$$R_{2k} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} \left(\frac{1-e^{it}}{2\pi i m - it} \right)^{2k} e^{-ikt} dt$$

$$= \frac{2^{2k}}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-(2m+1)\pi}^{-(2m-1)\pi} \left(\frac{\sin(\pi m + (\xi/2))}{\xi} \right)^{2k} d\xi$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t} \right)^{2k} dt = \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{\sin t}{t} \right)^{2k} dt.$$

The reasoning for odd $n = 2k+1, k \in \mathbb{N}$, is very similar. □

Now two main properties of the numbers R_n are contained in

Theorem 5.2.

i) *The sequence* (R_n) *is decreasing.*

ii) *The sequence* (R_n) *satisfies the complete asymptotic expansion*

$$R_n \approx \sqrt{\frac{6}{\pi n}} \left(1 + \sum_{\mu=1}^{\infty} \frac{q_{2\mu}}{n^{\mu}} \right), \tag{5.3}$$

as $n \to \infty$*, where*

$$q_{2\mu} = \sum_{k_2+2k_4+\ldots+\mu k_{2\mu}=\mu} (-6)^{\mu+s} \frac{(2\mu+2s)!}{(\mu+s)!} \prod_{r=1}^{\mu} \frac{1}{k_{2r}!} \left(\frac{\mathcal{B}_{2r+2}}{(2r+2)!(2r+2)} \right)^{k_{2r}}, \tag{5.4}$$

$n \in \mathbb{N},\ k_2 + k_4 + \ldots + k_{2\mu} = s,\ \mathcal{B}_\nu$ *being the Bernoulli numbers, and in particular*

$$R_n \sim \sqrt{\frac{6}{\pi n}} \left(1 - \frac{3}{20n} \right), \ n \to \infty.$$

In contrast to the central limit type asymptotics considered in sections 3 and 4 here (5.3) gives a complete asymptotic expansion in the usual sense of Poincaré [23]. Actually the decreasing property of R_n is proved in [2] for more general integrals of *sinc* type using an intricate analysis. Here for completeness for the special case given by Lemma 5.1, ii) we give a simple proof using a probabilistic argument. Also various asymptotics and numerical computations for R_n are known, e.g. [14], p.471, [21], [23], pp.94, 95. However, we state Theorem 5.2, ii), since all coefficients of the expansion (5.3) can be given explicitly by the expressions in (5.4). They result from Theorem 4.3 as a by-product.

Proof. i) We use the representation of R_n in Lemma 5.1, ii). Obviously we have $R_1 = R_2 = 1$. Starting with the density $p_1 := \frac{1}{2}\mathcal{X}_{[-1,1]}$ of the uniform distribution of the interval $[-1, 1]$ and its characteristic function

$$\varphi(t) = \frac{\sin t}{t} \ , \quad t \in \mathbb{R},$$

we consider the n-fold convolution

$$p_n := p_1 * \ldots * p_1$$

possessing the characteristic function

$$\varphi(t)^n = \left(\frac{\sin t}{t}\right)^n \ , \quad t \in \mathbb{R}, \ n \in \mathbb{N}.$$

By Fourier inversion we obtain

$$p_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \left(\frac{\sin t}{t}\right)^n dt \ , \quad n \geq 2, \ x \in \mathbb{R},$$

which, by Lemma 5.1, ii), implies that $R_n = 2p_n(0), n \geq 2$. Since the convolution of two symmetric densities each with a mode at 0 again is symmetric with mode at 0, see [6], p. 13, it follows that

$$R_{n+1} = 2p_{n+1}(0) = 2 \int_{-\infty}^{\infty} p_n(0 - x)p_1(x)dx \leq 2p_n(0) = R_n, \quad n \geq 2,$$

and part i) is proved.

ii) In view of Lemma 5.1, i) and Theorem 4.3 (note that $x = 0$ in both cases of Lemma 5.1, i)) immediately we get

$$R_n \approx \sqrt{\frac{6}{\pi n}} \left(1 + \sum_{\mu=1}^{\infty} \frac{\sqrt{2\pi}\ 12^\mu\ \tilde{q}_{2\mu}(0)}{n^\mu}\right)$$

as $n \to \infty$. Putting $q_{2\mu} := \sqrt{2\pi}\ 12^\mu\ \tilde{q}_{2\mu}(0),\ \mu \in \mathbb{N}$, and observing that

$$H_{2m}(0) = (-1)^m \frac{(2m)!}{m! 2^m} \ , \quad m \in \mathbb{N}_0,$$

see [24], p. 137, finally from (4.8) we obtain the expansion (5.3) with (5.4).

□

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