Binomial Polynomials

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Abstract The binomial polynomials are defined by the sum representation

$$Q_n^{(r)}(z) = \sum_{k=0}^{n} \binom{n}{k}^{r+1} z^k, \qquad n \in \mathbb{N},\ z \in \mathbb{C},$$

where r is a nonnegative integer. Using a multivariate complex integral representation the asymptotical behavior of the sequence $(Q_n^{(r)})_{n=1}^{\infty}$ is studied on the whole complex plane as $n \to \infty$. The proofs essentially are based on a multivariate version of the method of saddle points. Moreover, results on the asymptotic zero distribution for $(Q_n^{(r)})_{n=1}^{\infty}$ and for some related polynomials are established by means of the theory of logarithmic potentials with external fields. Classical results on sums of powers of binomial coefficients and for Legendre polynomials are generalized, as well as a specific weighted equilibrium problem on the unit interval is solved.

Keywords Asymptotics ; Asymptotic distribution of zeros ; Sums of binomial coefficients ; Binomial polynomials ; Weighted equilibrium problem

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1 Introduction

Sums of powers of binomial coefficients are of mathematical interest for a long time. Following a reference in [13], p. 201, it can be traced back to a work of C. Jordan from 1913 and many authors contributed to the subject. Works by Polya and Szegő ([13], p. 42), de Bruijn ([4], p. 72, p. 109), Comtet ([3], p. 90), Henrici ([6], p. 415) and Wong ([17], p. 505, p. 515) are concerned with asymptotics for sums of the particular form

$$S_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^{r+1} \quad \text{and} \quad A_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^{r+1} (-1)^k,$$

as $n \to \infty$, where r is a fixed nonnegative integer. It is known (see e.g. [6], p. 415) that we have

$$S_n^{(r)} \sim \frac{2^{(r+1)n}}{\sqrt{r+1}} \left(\frac{2}{\pi n}\right)^{r/2}, \tag{1.1}$$

as $n \to \infty$, and (see e.g. [4], p. 75)

$$A_{2n}^{(r)} \sim (-1)^n \frac{2^{2n(r+1)+1}}{(\pi n)^{r/2}\sqrt{r+1}} \left(\cos\left(\frac{\pi}{2(r+1)}\right)\right)^{2n(r+1)+r}, \tag{1.2}$$

as $n \to \infty$, meaning that the ratio of both sides tends to unity.

Results of this kind can be used, for example, to obtain informations about the existence of recurrence relations of a specific type. In general the behavior of sums of powers of binomial coefficients plays an important role in various parts of mathematics. In probability theory and statistics expressions of the form

$$B(n,p,r) = \sum_{k=0}^{n} \left(\binom{n}{k} p^k (1-p)^{n-k}\right)^{r+1},$$

where r is a nonnegative integer and $p \in (0,1)$, are studied in the context of asymptotic estimation theory in a work by Bowman and Shenton (see [1]). Actually their asymptotic result for $B(n,p,r)$ can be deduced from the local central limit theorem of de Moivre and Laplace for the binomial distribution (see, e.g., [5], chapter VII, 3).

In number theory sums of powers of binomial coefficients also play a considerable role. For instance, in a work by McIntosh (see [9]) the author studies the asymptotical behavior of sums of the form

$$\sum_{k=0}^{n} \binom{n}{k}^{r_0} \binom{n+k}{k}^{r_1} \cdot \ldots \cdot \binom{n+mk}{k}^{r_m},$$

where m and $r_0, \ldots, r_m$ are fixed nonnegative integers, extending a result on the sums

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

which was needed by Apéry in his famous proof of the irrationality of $\zeta(3)$. A further connection to number theory can be found in two recent papers by Noble (see [10], [11]), where the author studies sums of the form

$$\sum_{k=0}^{n} (-1)^{\epsilon k} \binom{n}{k} \binom{an}{k} d^k,$$

where $\epsilon \in \{0,1\}$ and a and d are positive integers, proving a conjecture by Chamberland and Dilcher.

In this paper we are concerned mainly with the sequence of the binomial polynomials

$$Q_n^{(r)}(z) = \sum_{k=0}^{n} \binom{n}{k}^{r+1} z^k, \qquad n \in \mathbb{N},\ z \in \mathbb{C},$$

where r is a fixed nonnegative integer. Considered from a complex perspective the behavior of the binomial polynomials on the complex plane as well as the behavior of its zeros is still unknown (except for the cases $r=0$ and $r=1$). In the case $r=1$ the polynomials $Q_n^{(1)}$ essentially are Legendre polynomials (see (1.5) below), for which various asymptotics are well known. In this sense the relations (1.1) and (1.2) are generalized on the one hand, while we extend classical results on the Legendre polynomials on the other hand. Further we mention the Bernstein operators of Bézier-type

$$\left(B_{n,\alpha} f\right)(x) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right)\left(J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)\right),$$

where $\alpha > 0$ and for $k \in \{0, \dots, n+1\}$

$$J_{n,k}(x) = \sum_{j=k}^{n} \binom{n}{j} x^j (1-x)^{n-j}.$$

These operators have been introduced by Chang in [2] who proved for continuous functions f on $[0,1]$ that $B_{n,\alpha} f$ converges to f uniformly on $[0,1]$ thereby generalizing the Weierstrass approximation theorem using Bernstein polynomials $B_{n,1}$. In this context for an investigation of finer approximation properties of the operators $B_{n,\alpha}$ the knowledge of asymptotics for the polynomials $Q_n^{(r)}$ might be useful (see also [2], [7], [8] and [18]).

The paper is structured as follows. In Section 2 we first prove an elementary inequality for a specific real function of several complex variables on the boundary of the unit circle (Lemma 2.1), which turns out to be a key ingredient for the proofs of the main results. In Lemma 2.2 we show that all zeros of the polynomials $Q_n^{(r)}$ are located on the negative real axis. The third section is the final section and contains the main results. Generalizing the result (1.1) in Theorem 3.1 we establish the approximation

$$Q_n^{(r)}(z) \sim \frac{\left(1 + z^{\frac{1}{r+1}}\right)^{(r+1)n+r}}{\left(2\pi n z^{\frac{1}{r+1}}\right)^{r/2} \sqrt{r+1}}, \qquad \text{as } n \to \infty, \tag{1.3}$$

valid for $z \in \mathbb{C}\backslash(-\infty, 0]$, where $z^{\frac{1}{r+1}} = \exp\left\{\frac{1}{r+1}(\log|z| + i\arg(z))\right\}$, $\arg(z) \in (-\pi, \pi)$. Moreover, extending the result (1.2) onto the negative real axis in Theorem 3.2 we obtain for fixed $r \geq 1$

$$
\begin{aligned}
Q_n^{(r)}(-x) =& \frac{2}{(2\pi n)^{r/2}\sqrt{r+1}} \left\{ \frac{\sin\left(\frac{\pi}{r+1}(1-\theta)\right)}{\sin\left(\frac{\pi}{r+1}\theta\right)} \right\}^{r/2} \qquad (1.4) \\
&\times \left\{ \frac{\sin\left(\frac{\pi}{r+1}\right)}{\sin\left(\frac{\pi}{r+1}(1-\theta)\right)} \right\}^{n(r+1)+r} \left(\cos\left(\pi n\theta + \frac{\pi r}{r+1}\left(\theta - \frac{1}{2}\right)\right) + o(1) \right),
\end{aligned}
$$

as $n \to \infty$, where we use for $x > 0$ the parametrization

$$
x = \left(\frac{\sin\left(\frac{\pi}{r+1}\theta\right)}{\sin\left(\frac{\pi}{r+1}(1-\theta)\right)} \right)^{r+1}, \qquad 0 < \theta < 1.
$$

The proofs will be carried out in the spirit of a multivariate version of the complex method of saddle points (for an explanation see the remarks previous to Theorem 3.1). We will not be concerned with the uniformity of those approximations on compact subsets of the range of their validity as it will not be needed for obtaining the desired results on the zeros. In order to characterize the behavior of the zeros of the polynomials $Q_n^{(r)}$, which are located on the negative real axis, it turns out to be convenient to translate the problem to the compact interval $[-1, 1]$. Thus we will also consider the polynomials

$$
P_n^{(r)}(z) = (z+1)^n Q_n^{(r)}\left(\frac{z-1}{z+1}\right) = \sum_{k=0}^{n} \binom{n}{k}^{r+1} (z-1)^k (z+1)^{n-k}. \tag{1.5}
$$

In this setting our results generalize classical results on the Legendre polynomials P_n, as in the case $r = 1$ we have the connection

$$
P_n^{(1)} = 2^n P_n,
$$

where P_n is defined in Szegő's classical monograph [16], chapter IV.

In the first part of Theorem 3.3 we show that the sequence of the normalized zero counting measures $(\nu_n)_n$ associated with $(P_n^{(r)})_n$ converges in the weak-star sense to a unit measure ν supported on $[-1, 1]$. This measure ν solves the weighted equilibrium problem on $[-1, 1]$ with respect to the weight function

$$
w(x) = |(x+1)^{\frac{1}{r+1}} + (x-1)^{\frac{1}{r+1}}|^{-(r+1)}, \tag{1.6}
$$

where $(x-1)^{\frac{1}{r+1}} = \exp\left(\frac{i\pi}{r+1}\right)(1-x)^{\frac{1}{r+1}}$ for $-1 \leq x \leq 1$, and its logarithmic potential is given by

$$\mathcal{U}^{\nu}(z) = -\log 2^{r+1} - \log |(z+1)^{\frac{1}{r+1}} + (z-1)^{\frac{1}{r+1}}|^{r+1}, \qquad z \in \mathbb{C}. \tag{1.7}$$

Moreover, the measure ν is absolutely continuous with respect to the Lebesgue measure on $[-1,1]$ and its Radon-Nikodym derivative is given by

$$\mathrm{d}\nu = \frac{\frac{2}{\pi}\sin\frac{\pi}{r+1}}{(x+1)^{1+\frac{1}{r+1}}(1-x)^{1-\frac{1}{r+1}} + 2(1-x^2)\cos\frac{\pi}{r+1} + (x+1)^{1-\frac{1}{r+1}}(1-x)^{1+\frac{1}{r+1}}}\,\mathrm{d}x. \tag{1.8}$$

In the second part of Theorem 3.3 we are concerned with the zeros of the binomial polynomials $Q_n^{(r)}$. We show that the sequence of the normalized zero counting measures $(\mu_n)_n$ associated with $(Q_n^{(r)})_n$ converges in the weak-star sense to a unit measure μ supported on $(-\infty, 0]$ whose logarithmic potential is given by

$$\mathcal{U}^{\mu}(z) = -(r+1)\log|1 + z^{\frac{1}{r+1}}|, \qquad z \in \mathbb{C}. \tag{1.9}$$

Moreover, the measure μ is absolutely continuous with respect to the Lebesgue measure on $(-\infty, 0]$ and its Radon-Nikodym derivative is given by

$$\mathrm{d}\mu = \frac{\sin\frac{\pi}{r+1}}{\pi}\;\frac{|x|^{\frac{1}{r+1}-1}}{1 + 2|x|^{\frac{1}{r+1}}\cos\frac{\pi}{r+1} + |x|^{\frac{2}{r+1}}}\,\mathrm{d}x. \tag{1.10}$$

Throughout the paper all potential theoretic notions are used as defined in the standard monograph [15].

2 Auxiliary Results

In this section we first prove an elementary inequality which we need for the proof of the main results.

Lemma 2.1. *Let $z \in \mathbb{C}\backslash(-\infty, 0]$ and $r \in \mathbb{N}$ be fixed numbers, $U = \partial\mathbb{D}$ the boundary of the complex unit disk, and let the function $H : U^r \to \mathbb{R}$ be defined by*

$$H(w_1, \dots, w_r) = \Re\left\{ z^{\frac{1}{r+1}} \left(r + 1 - \sum_{\nu=1}^{r} w_\nu - \prod_{\nu=1}^{r} w_\nu^{-1} \right) \right\},$$

where $z^{\frac{1}{r+1}} = \exp\left\{\frac{1}{r+1}(\log|z| + i\arg(z))\right\}$ with $\arg(z) \in (-\pi, \pi)$. Then we have

$$H(w_1, \dots, w_r) \geq 0$$

and $H(w_1, \dots, w_r) = 0$ if and only if $w_\nu = 1$ for all $\nu \in \{1, \dots, r\}$.

Proof. Without loss of generality we may assume that we have

$$z^{\frac{1}{r+1}} = e^{i\varphi}, \qquad \varphi \in \left(-\frac{\pi}{r+1}, \frac{\pi}{r+1}\right)$$

and

$$w_\nu = e^{ix_\nu}, \qquad x_\nu \in [0, 2\pi), \quad \nu \in \{1, \ldots, r\}.$$

Now we have to prove that the function $h : [0, 2\pi)^r \to \mathbb{R}$ defined by

$$h(x_1, \ldots, x_r) = \Re\left\{ e^{i\varphi} \left(r + 1 - \sum_{\nu=1}^{r} e^{ix_\nu} - \exp\left(-i \sum_{\nu=1}^{r} x_\nu \right) \right) \right\}$$

satisfies

$$h(x_1, \ldots, x_r) \geq 0$$

and $h(x_1, \ldots, x_r) = 0$ if and only if $x_\nu = 0$ for all $\nu \in \{1, \ldots, r\}$. In the case $r = 1$ the statement is true as we have $h(x_1) = 2\cos\varphi\,(1 - \cos x_1)$. Now let us assume that it is true for an integer $r - 1$ ($r \geq 2$). We have to study the function $h(x_1, \ldots, x_r)$ on $[0, 2\pi)^r$, where φ satisfies $-\frac{\pi}{r+1} < \varphi < \frac{\pi}{r+1}$. We consider two cases:

i) In the first case we consider points $(x_1, \ldots, x_r)$ that are located on the boundary of $[0, 2\pi)^r$. Hence, using symmetry and periodicity we can assume $x_r = 0$, so that we have

$$h(x_1, \ldots, x_{r-1}, 0) = \Re\left\{ e^{i\varphi} \left(r - \sum_{\nu=1}^{r-1} e^{ix_\nu} - \exp\left(-i \sum_{\nu=1}^{r-1} x_\nu \right) \right) \right\}.$$

By assumption it is clear that the last expression is positive in all points with exception of the origin, where it vanishes.

ii) In the second case we consider interior points of $[0, 2\pi)^r$. By an elementary calculation we obtain for all $j \in \{1, \ldots, r\}$

$$\frac{\partial h}{\partial x_j}(x_1, \ldots, x_r) = 2\sin\left(x_j + \frac{1}{2}\sum_{\nu \neq j} x_\nu \right) \cos\left(\frac{1}{2}\sum_{\nu \neq j} x_\nu - \varphi \right).$$

Let $(x_1^*, \ldots, x_r^*) \in (0, 2\pi)^r$ be a critical point, i.e. $\operatorname{grad} h(x_1^*, \ldots, x_r^*) = 0$. If one of the expressions

$$\cos\left(\frac{1}{2}\sum_{\nu \neq j} x_\nu^* - \varphi \right) \tag{2.1}$$

vanishes, then we have

$$\frac{\partial h}{\partial x_j}(x_1^*, \ldots, x_j, \ldots, x_r^*) = 0$$

for all $x_j \in (0, 2\pi)$. This means

$$h(x_1^*, \ldots, x_j^*, \ldots, x_r^*) = h(x_1^*, \ldots, 0, \ldots, x_r^*)$$

which is positive by assumption. Now we can assume that all expressions in (2.1) do not vanish so that we have for all $j \in \{1, \ldots, r\}$

$$\sin\left(x_j^* + \frac{1}{2}\sum_{\nu \neq j} x_\nu^*\right) = 0.$$

Hence, as we have $x_j^* \in (0, 2\pi)$, we obtain integers $k_1, \ldots, k_r \in \{1, \ldots, r\}$ such that

$$x_j^* + \frac{1}{2}\sum_{\nu \neq j} x_\nu^* = k_j \pi, \qquad j \in \{1, \ldots, r\}.$$

Let $x^* = (x_1^*, \ldots, x_r^*)^T$, $k = (k_1, \ldots, k_r)^T$ and let the matrix $A \in \mathbb{R}^{r,r}$ be defined by $A = (\frac{1}{2} + \frac{1}{2}\delta_{ij})$, where δ_{ij} denotes the Kronecker delta. Thus we have $Ax^* = \pi k$. It is easy to see that the inverse matrix of A is given by $A^{-1} = \frac{2}{r+1}(-1 + (r+1)\delta_{ij})$ so that we obtain for all $j \in \{1, \ldots, r\}$

$$x_j^* = \frac{2\pi}{r+1}\left(r k_j - \sum_{\nu \neq j} k_\nu\right).$$

Thus all expressions $\left(r k_j - \sum_{\nu \neq j} k_\nu\right)$ are integers contained in $\{1, \ldots, r\}$. Therefore for $j \in \{1, \ldots, r-1\}$ we have

$$(r+1)(k_j - k_{j+1}) \in \{-(r-1), \ldots, 0, \ldots, r-1\},$$

which only is possible if all numbers k_ν coincide. Hence, we can deduce that all numbers x_ν^* coincide and are given by $\frac{2\pi k}{r+1}$ for an integer $k \in \{1, \ldots, r\}$. So we merely need to study the function

$$g(t) = h(t, \ldots, t) = \Re\left\{e^{i\varphi}\left(r + 1 - re^{it} - e^{-irt}\right)\right\}$$

in the points $t = \frac{2\pi k}{r+1}$, $k \in \{1, \ldots, r\}$. Now we have

$$g\left(\frac{2\pi k}{r+1}\right) = 2(r+1)\sin\left(\frac{\pi k}{r+1}\right)\sin\left(\varphi + \frac{\pi k}{r+1}\right),$$

which obviously is positive for all $k \in \{1, \ldots, r\}$.

□

Remark 2.1. *By a slight modification of the proof of the previous lemma it is not difficult to treat the remaining case* $z \in (-\infty, 0)$. *This can be stated as follows: For fixed* $r \geq 2$ *and* $z \in (-\infty, 0)$ *we set* $z^{\frac{1}{r+1}} = e^{\frac{i\pi}{r+1}} |z|^{\frac{1}{r+1}}$. *Moreover, let* U *and the function* H *be defined as in Lemma 2.1. Then we have*

$$H(w_1, \ldots, w_r) \geq 0$$

and $H(w_1, \ldots, w_r) = 0$ *if and only if* $w_\nu = 1$ *for all* $\nu \in \{1, \ldots, r\}$ *or* $w_\nu = \exp\left(\frac{2\pi r i}{r+1}\right)$ *for all* $\nu \in \{1, \ldots, r\}$.

The next aim in this auxiliary section is a first result on the zeros of the polynomials $Q_n^{(r)}$.

Lemma 2.2. *All zeros of the binomial polynomials* $Q_n^{(r)}$ *are real and negative.*

Proof. We prove the statement by induction on r, using the following result from [14], p. 47, ex. 65: If all zeros of the real polynomial $\sum_{k=0}^{n} a_k z^k$ are real, then the same is true for the zeros of $\sum_{k=0}^{n} \frac{a_k}{k!} z^k$. Our statement clearly is true in the case $r = 0$. Now let us assume that it is true for an integer $r \geq 0$, which means that all zeros of the polynomials

$$Q_n^{(r)}(z) = \sum_{k=0}^{n} \binom{n}{k}^{r+1} z^k$$

are real and negative. Applying the above mentioned result from [14] yields that all zeros of the polynomials

$$\sum_{k=0}^{n} \binom{n}{k}^{r+1} \frac{z^k}{k!}$$

are real and thus they are negative. From this we can deduce that the same is true for the polynomials (in w)

$$\sum_{k=0}^{n} \binom{n}{k}^{r+1} \frac{w^k}{(n-k)!}.$$

Now from a further application of the above mentioned result from [14] we obtain that it is also true for the polynomials

$$\sum_{k=0}^{n} \binom{n}{k}^{r+1} \frac{w^k}{k!(n-k)!} = \frac{1}{n!} Q_n^{(r+1)}(w),$$

from which the statement follows. □

3 Main Results

Before we turn to the main results, as a preparation we state a suitable multivariate complex integral representation for the polynomials $Q_n^{(r)}$. This representation in fact is a generalization of the integral used in [6], p. 415, for treating the special case $z = 1$, i.e. for proving the result (1.1), and can be established in a similar manner.

Lemma 3.1. *For $z \in \mathbb{C}$ and $r, n \in \mathbb{N}$ we have*

$$Q_n^{(r)}(z) = \frac{1}{(2\pi i)^r} \int_\Gamma \left\{ \prod_{\nu=1}^r \left(1 + \frac{1}{w_\nu}\right) \left(1 + z \prod_{\nu=1}^r w_\nu\right) \right\}^n \prod_{\nu=1}^r w_\nu^{-1} \, \mathrm{d}w, \tag{3.1}$$

where $w = (w_1, \dots, w_r)$. This integral is considered to be an r-fold complex contour integral where Γ is given as a product of r positive oriented simple contours in the complex plane encircling the origin. Moreover, if $z^{\frac{1}{r+1}} = \exp\left\{\frac{1}{r+1}(\log|z| + i \arg(z))\right\}$, $\arg(z) \in (-\pi, \pi]$, *then using the parametrization*

$$w_j = z^{-\frac{1}{r+1}} e^{it_j}, \quad t_j \in [-\pi, \pi], \quad j \in \{1, \dots, r\},$$

we immediately obtain

$$Q_n^{(r)}(z) = \frac{1}{(2\pi)^r} \int_{[-\pi,\pi]^r} \left\{ \prod_{\nu=1}^r \left(1 + z^{\frac{1}{r+1}} e^{-it_\nu}\right) \left(1 + z^{\frac{1}{r+1}} \exp\left\{ i \sum_{\nu=1}^r t_\nu \right\}\right) \right\}^n \mathrm{d}t, \tag{3.2}$$

where $t = (t_1, \dots, t_r)$.

Proof. We observe for fixed z that the expression

$$\prod_{\nu=1}^r (1 + w_\nu)^n \left(1 + z \prod_{\nu=1}^r w_\nu\right)^n$$

can be considered as a holomorphic function of the variables $w_1, \dots, w_r$ defined on $\mathbb{C}^r$. Moreover, the coefficient of the term $\prod_{\nu=1}^r w_\nu^n$ of its multivariate Maclaurin expansion is given by $Q_n^{(r)}(z)$. Using Cauchy's integral formula yields

$$Q_n^{(r)}(z) = \frac{1}{(2\pi i)^r} \int_\Gamma \prod_{\nu=1}^r (1 + w_\nu)^n \left(1 + z \prod_{\nu=1}^r w_\nu\right)^n \prod_{\nu=1}^r w_\nu^{-n-1} \, \mathrm{d}w,$$

from which the statement follows. □

Next we turn to the first main result which deals with the asymptotical behavior of the binomial polynomials on the zero-free region $\mathbb{C} \setminus (-\infty, 0]$. The main difficulty in generalizing the treatments of the special cases $z = 1$ and $z = -1$ (see [6], p. 415, and [4], p. 72) is that in those cases it is possible to derive a real integral representation from (3.1). This enables one to invoke a multivariate generalization of the real Laplace method (as stated for instance in [6], p. 409) to obtain the asymptotic approximations (1.1) and (1.2). However, this property is lost in the general case for complex values of z. As it is known from the one-dimensional version of the Laplace method, the change from real integrals to complex integrals requires some essential modifications which lead to the method of saddle points (see, e.g [12], p. 125). Hence, the adequate tool for our proofs would be a complex multivariate generalization of the method of saddle points. Although a general and rigorous treatment of this multivariate complex case should be possible, it seems to be not present in the literature. As such a general treatment would not lie in the scope of this paper, our proofs we will be carried out in the spirit of such a method, meaning that we will use the necessary ideas and apply them in an ad-hoc way to the integral in Lemma 3.1. The task of choosing a set of paths that pass through a saddle point of the integrand is already solved by using the specific parametrization in (3.2).

Theorem 3.1. *Let r be a fixed positive integer and $z \in \mathbb{C}\backslash(-\infty, 0]$. Then we have*

$$Q_n^{(r)}(z) \sim \frac{\left(1 + z^{\frac{1}{r+1}}\right)^{(r+1)n+r}}{\left(2\pi n z^{\frac{1}{r+1}}\right)^{r/2} \sqrt{r+1}}\,, \qquad \textit{as} \quad n \to \infty,$$

where $z^{\frac{1}{r+1}} = \exp\left\{\frac{1}{r+1}(\log|z| + i\arg(z))\right\}$ *with* $\arg(z) \in (-\pi, \pi)$.

Proof. *i)* First we study the modulus of the integrand in the representation (3.2). We show ($t_\nu \in [-\pi, \pi]$):

$$\prod_{\nu=1}^{r} \left|1 + z^{\frac{1}{r+1}} e^{-it_\nu}\right| \left|1 + z^{\frac{1}{r+1}} \exp\left\{i \sum_{\nu=1}^{r} t_\nu\right\}\right| \leq |1 + z^{\frac{1}{r+1}}|^{r+1}, \tag{3.3}$$

where the inequality becomes an equality if and only if we have $t_\nu = 0$ for all $\nu \in \{1, \ldots, r\}$. To establish this result we first apply the inequality of the arithmetic and geometric mean and we obtain

$$\left(\prod_{\nu=1}^{r} \left|1 + z^{\frac{1}{r+1}} e^{-it_\nu}\right|^2 \left|1 + z^{\frac{1}{r+1}} \exp\left\{i \sum_{\nu=1}^{r} t_\nu\right\}\right|^2\right)^{\frac{1}{r+1}}$$
$$\leq \frac{1}{r+1} \sum_{\nu=1}^{r} \left|1 + z^{\frac{1}{r+1}} e^{-it_\nu}\right|^2 + \frac{1}{r+1} \left|1 + z^{\frac{1}{r+1}} \exp\left\{i \sum_{\nu=1}^{r} t_\nu\right\}\right|^2$$
$$= |1 + z^{\frac{1}{r+1}}|^2 - \frac{2}{r+1} \Re\left\{z^{\frac{1}{r+1}} \left(r + 1 - \sum_{\nu=1}^{r} e^{-it_\nu} - \exp\left(i \sum_{\nu=1}^{r} t_\nu\right)\right)\right\}.$$

Now an application of Lemma 2.1 easily establishes the result.

ii) Let $a(z) = \frac{z^{\frac{1}{r+1}}}{(1+z^{\frac{1}{r+1}})^2}$ for $z \in \mathbb{C}\backslash(-\infty, 0]$ and let the positive definite matrix $H \in \mathbb{R}^{r,r}$ be defined by $H = (1 + \delta_{ij})$, where δ_{ij} denotes the Kronecker delta. Using the estimate $\Re\left(a(z)\right) > 0$ it is not difficult to see that we have for every $\epsilon > 0$

$$\int\limits_{[-\epsilon,\epsilon]^r} e^{-\frac{n}{2}a(z)t^T Ht}\,\mathrm{d}t \sim \int\limits_{\mathbb{R}^r} e^{-\frac{n}{2}a(z)t^T Ht}\,\mathrm{d}t,$$

as $n \to \infty$, and the last integral can be evaluated explicitly by

$$\int\limits_{\mathbb{R}^r} e^{-\frac{n}{2}a(z)t^T Ht}\,\mathrm{d}t = \left(\frac{2\pi}{n}\right)^{r/2} \frac{(z^{\frac{1}{r+1}})^{-r/2}}{\sqrt{r+1}}(1 + z^{\frac{1}{r+1}})^r.$$

This evaluation can be performed by first showing the identity for positive values of z (in this case, by a real change of variables, the integral can be traced back to a standard integral, see, e.g., [6], p. 411), so that the general case follows by analytical continuation. Now we immediately obtain

$$\int\limits_{[-\epsilon,\epsilon]^r} e^{-\frac{n}{2}a(z)t^T Ht}\,\mathrm{d}t \sim \left(\frac{2\pi}{n}\right)^{r/2} \frac{(z^{\frac{1}{r+1}})^{-r/2}}{\sqrt{r+1}}(1 + z^{\frac{1}{r+1}})^r, \tag{3.4}$$

as $n \to \infty$.

iii) Next we show for small $\epsilon > 0$ the relation

$$\frac{1}{(2\pi)^r} \int\limits_{[-\epsilon,\epsilon]^r} e^{-np(t)}\,\mathrm{d}t \sim \frac{(1 + z^{\frac{1}{r+1}})^{(r+1)n}}{(2\pi)^r} \int\limits_{[-\epsilon,\epsilon]^r} e^{-\frac{n}{2}a(z)t^T Ht}\,\mathrm{d}t, \tag{3.5}$$

as $n \to \infty$, where the function $p(t) = p(t_1, \ldots, t_r)$ is defined by

$$p(t) = -\sum_{\nu=1}^{r} \log(1 + z^{\frac{1}{r+1}} e^{-it_\nu}) - \log\left(1 + z^{\frac{1}{r+1}} e^{i(t_1+\ldots+t_r)}\right).$$

This is equivalent to the relation

$$\int\limits_{[-\epsilon,\epsilon]^r} e^{-n(p(t)-p(0))}\,\mathrm{d}t \sim \int\limits_{[-\epsilon,\epsilon]^r} e^{-\frac{n}{2}a(z)t^T Ht}\,\mathrm{d}t,$$

as $n \to \infty$. According to (3.4) it is sufficient to show

$$\lim_{n\to\infty} n^{r/2} \left\{ \int\limits_{[-\epsilon,\epsilon]^r} e^{-n(p(t)-p(0))}\,\mathrm{d}t - \int\limits_{[-\epsilon,\epsilon]^r} e^{-\frac{n}{2}a(z)t^T Ht}\,\mathrm{d}t \right\} = 0.$$

As the function p is holomorphic in a neighbourhood of the origin it can be expanded in a complex multivariate Taylor series and we obtain ($t = (t_1 \ldots, t_r)^T$):

$$p(t) = p(0) + \operatorname{grad} p(0)^T t + \frac{1}{2} a(z) t^T H t + ||t||^2 R(t),$$

where $||t||$ denotes the Euclidean norm on $\mathbb{C}^r$ and $R(t) \to 0,\ t \to 0$. Calculating all complex partial derivatives shows $\frac{\partial p}{\partial t_\nu}(0) = 0$ for all $\nu \in \{1, \ldots, r\}$. Hence, in a small neighbourhood of the origin we have

$$p(t) = p(0) + \frac{1}{2} a(z) t^T H t + ||t||^2 R(t),$$

where $R(t) \to 0,\ t \to 0$. Now for small $\epsilon > 0$ we have

$$\begin{aligned}
&\left| n^{r/2} \left\{ \int\limits_{[-\epsilon,\epsilon]^r} e^{-n(p(t)-p(0))}\, \mathrm{d}t - \int\limits_{[-\epsilon,\epsilon]^r} e^{-\frac{n}{2} a(z) t^T H t}\, \mathrm{d}t \right\} \right| \\
&\leq \int\limits_{[-\epsilon,\epsilon]^r} n^{r/2} \left| e^{-n\left(\frac{1}{2} a(z) t^T H t + ||t||^2 R(t)\right)} - e^{-\frac{n}{2} a(z) t^T H t} \right| \mathrm{d}t \\
&= \int\limits_{[-\epsilon,\epsilon]^r} n^{r/2} e^{-\frac{n}{2} \Re(a(z)) t^T H t} \left| e^{-n\ ||t||^2 R(t)} - 1 \right| \mathrm{d}t.
\end{aligned}$$

Reminding $\Re(a(z)) > 0$ and $R(t) \to 0,\ t \to 0$, it is not difficult to show for small $\epsilon > 0$ that the limit of the last integral exists and vanishes for $n \to \infty$.

iv) Combining Lemma 3.1 and (3.3) we obtain for small $\epsilon > 0$

$$\begin{aligned}
Q_n^{(r)}(z) &= \frac{1}{(2\pi)^r} \int\limits_{[-\pi,\pi]^r} \left\{ \prod_{\nu=1}^{r} \left(1 + z^{\frac{1}{r+1}} e^{-it_\nu}\right) \left(1 + z^{\frac{1}{r+1}} \exp\left\{ i \sum_{\nu=1}^{r} t_\nu \right\}\right) \right\}^n \mathrm{d}t \\
&\sim \frac{1}{(2\pi)^r} \int\limits_{[-\epsilon,\epsilon]^r} e^{-np(t)}\, \mathrm{d}t, \qquad n \to \infty.
\end{aligned}$$

Using (3.4) and (3.5) we finally arrive at

$$\begin{aligned}
Q_n^{(r)}(z) &\sim \frac{(1 + z^{\frac{1}{r+1}})^{(r+1)n}}{(2\pi)^r} \int\limits_{[-\epsilon,\epsilon]^r} e^{-\frac{n}{2} a(z) t^T H t}\, \mathrm{d}t \\
&\sim \frac{\left(1 + z^{\frac{1}{r+1}}\right)^{(r+1)n+r}}{\left(2\pi n z^{\frac{1}{r+1}}\right)^{r/2} \sqrt{r+1}}, \qquad n \to \infty.
\end{aligned}$$

□

Next we will study the behavior of the polynomials $Q_n^{(r)}$ on the negative real axis expecting an oscillatory approximation in view of Lemma 2.2.

Theorem 3.2. *Let $r \geq 1$ be a fixed integer and $x > 0$. Then we have*

$$Q_n^{(r)}(-x) = \frac{2}{(2\pi n)^{r/2}\sqrt{r+1}} \left\{ \frac{\sin\left(\frac{\pi}{r+1}(1-\theta)\right)}{\sin\left(\frac{\pi}{r+1}\theta\right)} \right\}^{r/2}$$
$$\times \left\{ \frac{\sin\left(\frac{\pi}{r+1}\right)}{\sin\left(\frac{\pi}{r+1}(1-\theta)\right)} \right\}^{n(r+1)+r} \left(\cos\left(\pi n\theta + \frac{\pi r}{r+1}\left(\theta - \frac{1}{2}\right)\right) + o(1) \right),$$

as $n \to \infty$, where we use for $x > 0$ the parametrization

$$x = \left(\frac{\sin\left(\frac{\pi}{r+1}\theta\right)}{\sin\left(\frac{\pi}{r+1}(1-\theta)\right)} \right)^{r+1}, \qquad 0 < \theta < 1. \tag{3.6}$$

Proof. According to (1.5) the case $r = 1$ is known and can be deduced from results on the Legendre polynomials. Hence, we may assume that we have $r > 1$. Slightly modifying the parametrization used in (3.2) we obtain

$$\begin{aligned} Q_n^{(r)}(-x) &= \frac{1}{(2\pi)^r} \int\limits_{[-\pi,\pi]^r} \left\{ \prod_{\nu=1}^{r} \left(1 + x^{\frac{1}{r+1}} e^{-it_\nu}\right) \left(1 - x^{\frac{1}{r+1}} \exp\left\{ i \sum_{\nu=1}^{r} t_\nu \right\}\right) \right\}^n \mathrm{d}t \\ &= \frac{1}{(2\pi)^r} \int\limits_{[-\pi,\pi]^r} e^{-np(t)}\, \mathrm{d}t, \end{aligned} \tag{3.7}$$

with an obvious definition of the function p. Using Lemma 2.1 and its following remark it is not difficult to see that the modulus of the integrand in (3.7) possesses exactly two maxima on the range of integration which are located at the points $t^{(1)} = \left(\frac{\pi}{r+1}, \dots, \frac{\pi}{r+1}\right)$ and $t^{(2)} = \left(-\frac{\pi}{r+1}, \dots, -\frac{\pi}{r+1}\right)$. Moreover, a calculation shows that the complex gradient of p vanishes in both points $t^{(1)}$ and $t^{(2)}$. For $x > 0$ let the function $b(x)$ be defined by

$$b(x) = \frac{x^{\frac{1}{r+1}} e^{-i\frac{\pi}{r+1}}}{(1 + x^{\frac{1}{r+1}} e^{-i\frac{\pi}{r+1}})^2}.$$

Then the Hessians of p in the points $t^{(1)}$ and $t^{(2)}$ are given by

$$\operatorname{Hess} p(t^{(1)}) = b(x)\, H, \qquad \operatorname{Hess} p(t^{(2)}) = \overline{b(x)}\, H,$$

where the matrix H is defined in the proof of Theorem 3.1. Splitting the integral in (3.7) into two parts yields

$$Q_n^{(r)}(-x) = I_n^{(1)}(x) + I_n^{(2)}(x),$$

where

$$I_n^{(1)}(x) = \frac{1}{(2\pi)^r} \int\limits_{E_+} e^{-np(t)}\, \mathrm{d}t, \qquad I_n^{(2)}(x) = \frac{1}{(2\pi)^r} \int\limits_{E_-} e^{-np(t)}\, \mathrm{d}t,$$

with

$$E_+ = \left\{ (\xi_1, \dots, \xi_r)^T \in [-\pi, \pi]^r \mid \sum_{\nu=1}^{r} \xi_\nu > 0 \right\},$$

and

$$E_- = \left\{ (\xi_1, \dots, \xi_r)^T \in [-\pi, \pi]^r \mid \sum_{\nu=1}^{r} \xi_\nu < 0 \right\}.$$

Now both of the integrals $I_n^{(1)}(x)$ and $I_n^{(2)}(x)$ can be evaluated asymptotically using the same method applied in the proof of Theorem 3.1 so that we obtain

$$I_n^{(1)}(x) = \frac{x^{-\frac{r}{2(r+1)}} e^{\frac{\pi r i}{2(r+1)}}}{(2\pi n)^{r/2}\sqrt{r+1}} \left(1 + x^{\frac{1}{r+1}} e^{-\frac{\pi i}{r+1}}\right)^{(r+1)n+r} (1 + o(1))$$

and

$$I_n^{(2)}(x) = \frac{x^{-\frac{r}{2(r+1)}} e^{\frac{-\pi r i}{2(r+1)}}}{(2\pi n)^{r/2}\sqrt{r+1}} \left(1 + x^{\frac{1}{r+1}} e^{\frac{\pi i}{r+1}}\right)^{(r+1)n+r} (1 + o(1)),$$

as $n \to \infty$. Let the function $G_n(x)$ be defined by

$$G_n(x) = \frac{x^{-\frac{r}{2(r+1)}} e^{\frac{\pi r i}{2(r+1)}}}{(2\pi n)^{r/2}\sqrt{r+1}} \left(1 + x^{\frac{1}{r+1}} e^{-\frac{\pi i}{r+1}}\right)^{(r+1)n+r},$$

then we clearly have for $n \to \infty$

$$Q_n^{(r)}(-x) = G_n(x)(1 + o(1)) + \overline{G_n(x)}(1 + o(1)).$$

Now using the parametrization given in (3.6) an elementary calculation yields

$$\begin{aligned} Q_n^{(r)}(-x) =& \frac{1}{(2\pi n)^{r/2}\sqrt{r+1}} \left\{ \frac{\sin\left(\frac{\pi}{r+1}(1-\theta)\right)}{\sin\left(\frac{\pi}{r+1}\theta\right)} \right\}^{r/2} \left\{ \frac{\sin\left(\frac{\pi}{r+1}\right)}{\sin\left(\frac{\pi}{r+1}(1-\theta)\right)} \right\}^{n(r+1)+r} \\ & \times \left\{ a_n(1 + o(1)) + \overline{a_n}(1 + o(1)) \right\}, \quad n \to \infty, \end{aligned}$$

where $a_n = \exp\left(i\left(\frac{\pi r}{2(r+1)} - \frac{\pi}{r+1}\theta(n(r+1)+r)\right)\right)$. From this the claimed result follows immediately. □

Our next aim is to study the behavior of the zeros of the polynomials $Q_n^{(r)}$ using tools from complex potential theory. As explained in the introduction it turns out to be convenient to study first the zeros of the polynomials $P_n^{(r)}$ as defined in (1.5). Having in mind that the complex transformation $T(x) = \frac{x-1}{x+1}$ maps the interval $(-1, 1)$ bijectively onto the negative real axis $(-\infty, 0)$ we can conclude from Lemma 2.2 that all zeros of the polynomials $P_n^{(r)}$ are located in $(-1, 1)$.

Theorem 3.3. *Let the $(\nu_n)_n$ denote the sequence of the normalized zero counting measures associated with $(P_n^{(r)})_n$ and let $(\mu_n)_n$ the sequence of the normalized zero counting measures associated with $(Q_n^{(r)})_n$. Then we have:*

i) The sequence $(\nu_n)_n$ converges in the weak-star sense to a unit measure ν supported on $[-1,1]$. This measure ν solves the weighted equilibrium problem on $[-1,1]$ with respect to the weight function

$$w(x) = |(x+1)^{\frac{1}{r+1}} + (x-1)^{\frac{1}{r+1}}|^{-(r+1)},$$

where $(x-1)^{\frac{1}{r+1}} = \exp\left(\frac{i\pi}{r+1}\right)(1-x)^{\frac{1}{r+1}}$ for $-1 \le x \le 1$, and its logarithmic potential is given by

$$\mathcal{U}^{\nu}(z) = -\log 2^{r+1} - \log|(z+1)^{\frac{1}{r+1}} + (z-1)^{\frac{1}{r+1}}|^{r+1}, \qquad z \in \mathbb{C}.$$

Moreover, the measure ν is absolutely continuous with respect to the Lebesgue measure on $[-1,1]$ and its Radon-Nikodym derivative is given by

$$\mathrm{d}\nu = \frac{\frac{2}{\pi}\sin\frac{\pi}{r+1}}{(x+1)^{1+\frac{1}{r+1}}(1-x)^{1-\frac{1}{r+1}} + 2(1-x^2)\cos\frac{\pi}{r+1} + (x+1)^{1-\frac{1}{r+1}}(1-x)^{1+\frac{1}{r+1}}}\,\mathrm{d}x.$$

ii) The sequence $(\mu_n)_n$ converges in the weak-star sense to a unit measure μ supported on $(-\infty,0]$. Its logarithmic potential is given by

$$\mathcal{U}^{\mu}(z) = -(r+1)\log|1 + z^{\frac{1}{r+1}}|, \qquad z \in \mathbb{C}.$$

Moreover, the measure μ is absolutely continuous with respect to the Lebesgue measure on $(-\infty,0]$ and its Radon-Nikodym derivative is given by

$$\mathrm{d}\mu = \frac{\sin\frac{\pi}{r+1}}{\pi}\,\frac{|x|^{\frac{1}{r+1}-1}}{1 + 2|x|^{\frac{1}{r+1}}\cos\frac{\pi}{r+1} + |x|^{\frac{2}{r+1}}}\,\mathrm{d}x.$$

Proof. *i)* First we observe that the logarithmic potentials of the measures ν_n can be expressed by using the polynomials $Q_n^{(r)}$:

$$\begin{aligned}\mathcal{U}^{\nu_n}(z) &= \int \log|z-t|^{-1}\,\mathrm{d}\nu_n(t)\\ &= -\log|Q_n^{(r)}(1)|^{1/n} - \log\left(|z+1|\left|Q_n^{(r)}\left(\frac{z-1}{z+1}\right)\right|^{1/n}\right).\end{aligned}$$

For $z \in \mathbb{C}\backslash[-1,1]$ we therefore obtain by an application of Theorem 3.1

$$\lim_{n\to\infty}\mathcal{U}^{\nu_n}(z) = -\log 2^{r+1} - \log|(z+1)^{\frac{1}{r+1}} + (z-1)^{\frac{1}{r+1}}|^{r+1}. \tag{3.8}$$

Hence, using Helly's selection principle ([15], p. 3) and Carleson's unicity theorem ([15], p. 123) we can conclude that the sequence (ν_n) converges in the weak-star sense to a unit measure ν supported on $[-1,1]$ and its logarithmic potential $\mathcal{U}^\nu$ is given by the right side of the equation (3.8) on $\mathbb{C}\backslash[-1,1]$. Exploiting that $\mathcal{U}^\nu$ and the right side of the equation (3.8) is continuous on $\mathbb{C}$ with respect to the fine topology (see, e.g. [15], p. 58) und using that the boundary of $\mathbb{C}\backslash[-1,1]$ in the fine topology coincides with its boundary in the Euclidean topology (see [15], p. 61) we obtain

$$\mathcal{U}^\nu(z) = -\log 2^{r+1} - \log|(z+1)^{\frac{1}{r+1}} + (z-1)^{\frac{1}{r+1}}|^{r+1}, \quad z \in \mathbb{C}.$$

As the measure ν clearly has finite logarithmic energy and as we have

$$\mathcal{U}^\nu(z) + \log|(z+1)^{\frac{1}{r+1}} + (z-1)^{\frac{1}{r+1}}|^{r+1} = -\log 2^{r+1}, \quad z \in [-1,1],$$

we can deduce from Remark 1.5 in [15], p. 28, that ν solves the equilibrium problem on $[-1,1]$ with respect to the weight function $w(x)$ defined in (1.6) and its modified Robin constant is given by $-\log 2^{r+1}$. In order to determine the Radon-Nikodym derivative of ν using Carleson's unicity theorem it is sufficient to verify

$$\begin{aligned}\int_{-1}^{1} &\frac{\frac{2}{\pi}\sin\frac{\pi}{r+1}\,\log|z-x|^{-1}}{(x+1)^{1+\frac{1}{r+1}}(1-x)^{1-\frac{1}{r+1}} + 2(1-x^2)\cos\frac{\pi}{r+1} + (x+1)^{1-\frac{1}{r+1}}(1-x)^{1+\frac{1}{r+1}}}\,\mathrm{d}x \\ = \; &-\log 2^{r+1} - \log|(z+1)^{\frac{1}{r+1}} + (z-1)^{\frac{1}{r+1}}|^{r+1}, \quad z \in \mathbb{C}\backslash[-1,1].\end{aligned}$$

This identity can be established, after a change of variables, by considering the integral

$$\frac{\sin\frac{\pi}{r+1}}{\pi}\int_0^\infty \frac{y^{\frac{1}{r+1}-1}}{(1+y^{\frac{1}{r+1}}e^{\frac{i\pi}{r+1}})(1+y^{\frac{1}{r+1}}e^{\frac{-i\pi}{r+1}})(z-\frac{y-1}{y+1})}\,\mathrm{d}y,$$

which can be evaluated by an application of residue calculus to the complex contour integral

$$\int_\gamma \frac{\zeta^{\frac{1}{r+1}-1}}{1+\zeta^{\frac{1}{r+1}}}\,\frac{1}{z-\frac{\zeta+1}{\zeta-1}}\,\mathrm{d}\zeta,$$

where γ is a simple positive oriented contour in $\mathbb{C}\backslash(-\infty,0]$ encircling the point $\zeta = \frac{z+1}{z-1}$ and suitably approximating the negative real axis from both sides.

ii) We can consider the measures μ_n to be the images of the measures ν_n with respect to the transformation $T(x) = \frac{x-1}{x+1}$, i.e. $\mu_n = \nu_n^T$. From the weak-star convergence of the sequence $(\nu_n)_n$ to ν it follows that the sequence

$(\mu_n)_n$ converges in the weak-star sense to the measure $\mu = \nu^T$. Using this connection yields for $z \in \mathbb{C}$

$$\begin{aligned} \mathcal{U}^\mu(z) &= \mathcal{U}^\nu\left(\frac{z+1}{z-1}\right) - \mathcal{U}^\nu(-1) - \log|z-1| \\ &= -(r+1)\log|1+z^{\frac{1}{r+1}}|. \end{aligned}$$

Moreover, again using $\mu = \nu^T$ the claimed Radon-Nikodym derivative for μ can be obtained from those for ν by an easy calculation.

□

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