Noname manuscript No.
(will be inserted by the editor)

Phased unitary Golay pairs, Butson Hadamard matrices and a conjecture of Ito's

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the date of receipt and acceptance should be inserted later

Abstract Pairs of complementary binary or quaternary sequences of length v such as Golay pairs, complex Golay pairs and periodic Golay pairs may be used to construct Hadamard matrices and complex Hadamard matrices of order $2v$. We generalize these and define unitary Golay pairs and phased unitary Golay pairs of length v with entries in the k^{th} roots of unity for any $k \geq 2$. This leads to a construction of Butson Hadamard matrices of order $2v$ over the k^{th} roots of unity for even k. Ito conjectured that a central relative $(4v, 2, 4v, 2v)$-difference set exists in a dicyclic group of order $8v$ for all $v \geq 1$, and this is known to imply the Hadamard conjecture. With our construction we prove that Ito's conjecture also implies the stronger complex Hadamard conjecture. As a consequence, with this method we construct a complex Hadamard matrix of order $2v$ for any v for which Ito's conjecture is verified, in particular, any $v \leq 46$.

Keywords Unitary Golay pair, Butson Hadamard matrix, complementary sequences

Mathematics Subject Classification (2000) 05B20, 11B99, 15B33

1 Introduction

Let k be a positive integer and let $\zeta_k = e^{\frac{2\pi\sqrt{-1}}{k}}$. A $n \times n$ matrix H with entries in $\{\zeta_k^i\}_{0 \leq i < k}$ is *Butson Hadamard* [5] $\mathrm{BH}(n, k)$ if $HH^* = nI_n$ where H^* denotes the conjugate transpose of H. A $\mathrm{BH}(n, 2)$ is called a Hadamard matrix and a $\mathrm{BH}(n, 4)$ is often called a complex Hadamard matrix. Applications of Butson Hadamard matrices range from quantum information theory to coding and cryptography, and thus constructions of these matrices are highly valuable. They play a key role in quantum teleportation [24], a promising secure method of communication. We refer to [22] for a detailed guide to Butson Hadamard matrices, and also to an up to date online catalogue at [4]. Some successful constructions of

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Hadamard and complex Hadamard matrices rely on the existence of complementary binary and quaternary sequences. These are sets, usually pairs, of sequences with autocorrelation at each phase summing to zero. Sequences with desirable autocorrelation properties have a wide range of applications similar to Butson Hadamard matrices, typically in branches of modern communications. Construction of Butson Hadamard matrices is our primary motivation in writing this paper, but we bear in mind the potential for wider applications. All computational procedures in this paper have been implemented in MAGMA [3]; we test equivalence of Butson Hadamard matrices using the procedure of [13, Section 3].

2 Preliminaries

The *complex aperiodic autocorrelation function* of a $\mathbb{C}$-sequence a of length v and shift s is defined to be

$$\mathrm{CAF}_s(a) = \textstyle\sum_{i=0}^{v-s-1} a_i \overline{a_{i+s}}$$

where $\overline{z}$ denotes the complex conjugate of z, and a is indexed beginning at 0. A *complex Golay pair* ($\mathrm{CGP}(v)$) is a pair of 4-ary sequences (a, b) of length v such that $\mathrm{CAF}_s(a) + \mathrm{CAF}_s(b) = 0$ for all $1 \leq s \leq v-1$. In the special case of binary sequences, we obtain *Golay pairs* [15] of length v ($\mathrm{GP}(v)$). Turyn [23] proved that $\mathrm{GP}(v) \neq \varnothing$ for $v = 2^a 10^b 26^c$ for all $a, b, c \geq 0$, however no others are known to exist. Defined and developed in [7,8], $\mathrm{CGP}(v)$ is non-empty for all $v = 2^{x+u} 3^y 5^c 11^d 13^e$ where $x, y, c, d, e, u \geq 0$, $y+c+d+e \leq x+2u+1$ and $u \leq c + e$. We define a *unitary Golay pair* ($\mathrm{UGP}(v, k)$) to be a pair of k-ary sequences (a, b) of length v such that $\mathrm{CAF}_s(a) + \mathrm{CAF}_s(b) = 0$ for all $1 \leq s \leq v - 1$. It follows that $\mathrm{UGP}(v, k)$ is non-empty only if k is even when $v > 1$.

To some extent the idea of complex Golay pairs has already been generalized to k-ary sequences, see [14,21], where in particular some 4-ary and 6-ary pairs are constructed. Other authors [1,18] also study autocorrelation properties of individual k-ary sequences. In [9] complementary 2^h-ary sequences of length 2^n are constructed for positive integers h, n, and an interesting connection between these sequences and Reed-Muller codes is proven. This was generalized to q-ary sequences for even q by Paterson in [19]. Kamali and Kharaghani [17] introduce dihedral Golay sequences which also generalize complex Golay sequences. An exposition on this topic requires a discussion on signed groups, which is a digression from the focus of this paper. We note that sequences with entries in signed groups may be used to construct signed group Hadamard matrices [6], and the main ideas presented in this paper can be extended to incorporate sequences with entries in an abelian signed group.

Throughout this paper, let

$$C_{k,m} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & & 0 & 1 \\ \zeta_k^m & 0 & 0 & & 0 & 0 \end{bmatrix}. \tag{1}$$

We define the *phased periodic autocorrelation function* of a k-ary sequence a of length v and shift s by $\mathrm{PPAF}_{m,s}(a) = a \cdot \overline{aC^s_{k,m}}$, treating a as a row vector. This generalizes both the periodic ($m = 0$) and negaperiodic ($m = 1$) autocorrelation functions defined for binary sequences. We say (a, b) is a *phased unitary Golay pair* ($\mathrm{PUGP}(v, k, m)$) if $\mathrm{PPAF}_{m,s}(a) + \mathrm{PPAF}_{m,s}(b) = 0$ for all $1 \leq s \leq v - 1$. It is not necessary that k is even for $\mathrm{PUGP}(v, k, m)$ to be non-empty.

A *periodic Golay pair* ($\mathrm{PGP}(v)$) is a $\mathrm{PUGP}(v, 2, 0)$ and a *negaperiodic Golay pair* ($\mathrm{NGP}(v)$) is a $\mathrm{PUGP}(v, 2, 1)$. Both of these generalize Golay pairs; it is shown in [2] that $\mathrm{GP}(v) = \mathrm{PGP}(v) \cap \mathrm{NGP}(v)$. Both $\mathrm{PGP}(v)$ and $\mathrm{NGP}(v)$ are non-empty only if v is even, and in the case of the former, only if v is the sum of two squares. At the time of writing, the smallest v for which existence of a $\mathrm{PGP}(v)$ is undecided is 90 [11]. The term NGP first appears in [2], though NGPs are precisely the *associated pairs* defined by Ito [16] years prior. Currently there is no even v known such that $\mathrm{NGP}(v) = \varnothing$; Ito conjectured that there is no such v. The smallest length for which existence is currently undecided is 94, see [20]. At least one associated pair of length $2n$ for $n \leq 45$ is listed in [16]. The present author compiled complete enumerations of $\mathrm{NGP}(2n)$ for all $n \leq 10$ in [12].

Let a and b be k-ary sequences. Then $\overline{a}$ is the sequence obtained by replacing each entry of a with its complex conjugate; $\hat{a}$ is the sequence obtained by reversing a; and $a^* = \hat{\overline{a}} = \overline{\hat{a}}$.

Lemma 2.1 *Let* $(a, b) \in \mathrm{PUGP}(v, k, m)$. *Then*

1. $(a^*, b^*) \in \mathrm{PUGP}(v, k, m)$,
2. $(\hat{a}, \hat{b}) \in \mathrm{PUGP}(v, k, k - m)$,
3. $(\overline{a}, \overline{b}) \in \mathrm{PUGP}(v, k, k - m)$.

Hereafter we use the shorthand notation $\mathrm{CAF}_s(a, b) := \mathrm{CAF}_s(a) + \mathrm{CAF}_s(b)$ and $\mathrm{PPAF}_{m,s}(a, b) := \mathrm{PPAF}_{m,s}(a) + \mathrm{PPAF}_{m,s}(b)$.

Theorem 2.2 $\mathrm{UGP}(v, k) = \cap_{m=0}^{k-1} \mathrm{PUGP}(v, k, m)$.

Proof Suppose that $(a, b) \in \mathrm{UGP}(v, k)$. Then for any $m < k$ and $s < v$,

$$\mathrm{PPAF}_{m,s}(a, b) = \mathrm{CAF}_s(\hat{a}, \hat{b}) + \zeta_k^{-m} \mathrm{CAF}_{v-s}(a, b) = 0.$$

Thus $(a, b) \in \cap_{m=0}^{k-1} \mathrm{PUGP}(v, k, m)$.

Now suppose that $(a, b) \in \cap_{m=0}^{k-1} \mathrm{PUGP}(v, k, m)$. Then for any $1 \leq s \leq v-1$ and $m_1 \neq m_2$ we get $\zeta_k^{-m_1} (\sum_{i=0}^{s-1} a_i \overline{a_{i+v-s}} + b_i \overline{b_{i+v-s}}) = \zeta_k^{-m_2} (\sum_{i=0}^{s-1} a_i \overline{a_{i+v-s}} + b_i \overline{b_{i+v-s}})$ implying that $\mathrm{CAF}_{v-s}(a, b) = 0$ for any $1 \leq s \leq v - 1$, proving the result.

Corollary 2.3 *If* $(a, b) \in \mathrm{PUGP}(v, k, m_1) \cap \mathrm{PUGP}(v, k, m_2)$ *for any* $m_1 \neq m_2$ *then* $(a, b) \in \mathrm{UGP}(v, k)$.

Corollary 2.4 *Suppose* $m \notin \{0, \frac{k}{2}\}$. *If* a *and* b *are symmetric and* $(a, b) \in \mathrm{PUGP}(v, k, m)$, *then* $(a, b) \in \mathrm{UGP}(v, k)$.

3 Constructing Butson Hadamard matrices

Denote the set of monomial $n \times n$ matrices with non-zero entries in a set K by $\mathrm{Mon}(n, K)$, and let $\langle \zeta_k \rangle$ denote the set of elements of the cyclic group generated by ζ_k. Let H and H' be $\mathrm{BH}(n, k)$. We say $H \approx H'$ are *equivalent* if there exists $(P, Q) \in \mathrm{Mon}(n, \langle \zeta_k \rangle)^2$ such that $PHQ^* = H'$. We will say that a $\mathrm{BH}(n, k)$ is *proper* if it is not equivalent to any $\mathrm{BH}(n, k')$ for all $k' < k$.

Proposition 3.1 *Let* $(a, b) \in \mathrm{PUGP}(v, k, m)$. *Define* A *and* B *to be* $v \times v$ *matrices with first row* a *and* b *respectively, where every subsequent row is given by* $A_{i+1} = A_i C_{k,m}$ *and* $B_{i+1} = B_i C_{k,m}$. *Then* $AB = BA$.

Theorem 3.2 *Assuming the notation of Proposition 3.1, the matrix*

$$H = \begin{bmatrix} A & B \\ -B^* & A^* \end{bmatrix}$$

is $\mathrm{BH}(2v, k)$ *if* k *is even and* $\mathrm{BH}(2v, 2k)$ *if* k *is odd.*

By Theorem 2.2, given an element of $\mathrm{UGP}(v, k)$ we can construct k possibly inequivalent $\mathrm{BH}(2v, k)$. Let k be even for the remainder of this section.

Remark 3.3 Noting that $\mathrm{GP}(v) \subseteq \mathrm{UGP}(v, k)$, any element of $\mathrm{GP}(v)$ can be used to construct up to k inequivalent $\mathrm{BH}(2v, k)$. Depending on the choice of m in the construction above the matrix constructed may be proper. In particular, the matrix will be proper if m is coprime to k.

Theorem 3.4 *Let* $n = 2^a 10^b 26^c$ *where* $a, b, c \geq 0$, *and* $n \geq 2$. *Then there exists a proper* $\mathrm{BH}(2n, k)$ *for all even* k.

More generally, $\mathrm{UGP}(v, k) \subseteq \mathrm{UGP}(v, nk)$ for any $n \geq 1$, and thus given $(a, b) \in \mathrm{UGP}(v, k)$ it is possible to construct up to nk inequivalent $\mathrm{BH}(2v, nk)$, some of which are proper.

Theorem 3.5 *Suppose that* $\mathrm{UGP}(v, k) \neq \varnothing$. *Then* $\mathrm{UGP}(v, nk) \neq \varnothing$ *for all* $n \geq 1$, *and consequently there exists a proper* $\mathrm{BH}(2v, nk)$ *for all* $n \geq 1$.

4 Ito's conjecture

Ito [16] conjectured that $\mathrm{PUGP}(2v, 2, 1)$ is non-empty for all $v \geq 1$. We will now describe a bijection between $\mathrm{PUGP}(v, 4, 1)$ and $\mathrm{PUGP}(2v, 2, 1)$, and thus Ito's conjecture implies the complex Hadamard conjecture through our construction above.

Let $i = \sqrt{-1}$. We will require the following.

Proposition 4.1 *Let* a *be a* 4*-ary sequence of length* v. *Then* $a \cdot \overline{aC^s_{4,1}} = -i(\overline{a \cdot \overline{aC^{v-s}_{4,1}}})$.

Let X be the set of 4-ary sequences of length v and Y be the set of 2-ary sequences of length $2v$. Now let $f : X \to Y$ be the bijective map such that $f(x) = y$ where y is obtained from $x \circ \overline{x}$, the concatenation of x and $\overline{x}$, by replacing each i with -1, and each $-i$ with 1. Observe that $f^{-1} : Y \to X$ is realised by $f^{-1}(y) = x$ where $x_j = 1, -1, i$ or $-i$ if $[y_j, y_{j+v}] = [1, 1], [-1, -1], [-1, 1]$, or $[1, -1]$ respectively. Let $\mathrm{Re}(z)$ denote the real part of $z \in \mathbb{C}$.

Proposition 4.2 *Let* $a, b \in X$*. Then* $\mathrm{PPAF}_{1,s}(a, b) = 0$ *if and only if* $\mathrm{PPAF}_{1,s}(f(a), f(b)) = 0$ *for any* $s \in \{1, \ldots, v-1\}$.

Proof Consider the terms in the sum $a \cdot \overline{aC^s_{4,1}}$. If $t \geq s$, given the term $a_t\overline{a_{t-s}}$ we may determine the value of $f(a)_t\overline{f(a)_{t-s}} + f(a)_{v+t}\overline{f(a)_{v+t-s}}$. Similarly, if $t < s$, given the term $-ia_t\overline{a_{v+t-s}}$ we may determine the value of $-f(a)_t\overline{f(a)_{2v+t-s}} + f(a)_{v+t}\overline{f(a)_{v+t-s}}$. Each term equal to 1 in $a \cdot \overline{aC^s_{4,1}}$ corresponds to two terms equal to 1 in $f(a) \cdot \overline{f(a)C^s_{2,1}}$, and each term equal to -1 corresponds to two terms equal to -1 in $f(a) \cdot \overline{f(a)C^s_{2,1}}$. Each term equal to $\pm i$ corresponds to two terms that sum to zero in $f(a) \cdot \overline{f(a)C^s_{2,1}}$. Thus if $\mathrm{PPAF}_{1,s}(a, b) = 0$, then $\mathrm{PPAF}_{1,s}(f(a), f(b)) = 0$. In the other direction, the reverse of this argument proves that if $\mathrm{PPAF}_{1,s}(f(a), f(b)) = 0$, then $\mathrm{Re}(\mathrm{PPAF}_{1,s}(a, b)) = 0$. Finally, if $\mathrm{PPAF}_{1,v-s}(f(a), f(b)) = 0$ then $\mathrm{Re}(\mathrm{PPAF}_{1,v-s}(a, b)) = 0$ and Proposition 4.1 completes the proof.

Theorem 4.3 *Define the bijection* $\phi : X \times X \to Y \times Y$ *where* $\phi(a, b) = (f(a), f(b))$ *for all* $a, b \in X$*. The restriction of* ϕ *to* $\mathrm{PUGP}(v, 4, 1)$ *is a bijection from* $\mathrm{PUGP}(v, 4, 1)$ *into* $\mathrm{PUGP}(2v, 2, 1)$.

Proof Let $(a, b) \in \mathrm{PUGP}(v, 4, 1)$. By Proposition 4.2 $\mathrm{PPAF}_{1,s}(a, b) = 0$ if and only if $\mathrm{PPAF}_{1,s}(f(a), f(b)) = 0$ for all $1 \leq s \leq v-1$. Further, $f(a) \cdot \overline{f(a)C^v_{2,1}} = 0$. The result follows.

Hence the existence of a $\mathrm{BH}(2v, 4)$ is confirmed for all v such that $\mathrm{PUGP}(2v, 2, 1)$ is known to be non-empty. By [12, Theorem 3], $\mathrm{PUGP}(2v, 2, 1) \neq \varnothing$ if and only if there a central relative $(4v, 2, 4v, 2v)$-difference set in Q_{8v}, the dicyclic group of order $8v$. Schmidt [20, Corollary 3.6] proves the following.

Theorem 4.4 *Let* m *be a positive integer such that* $2m-1$ *and* $4m-1$ *is a prime power or* m *is odd and there is a Williamson matrix over* $\mathbb{Z}_m$*. Then there is a relative* $(4t, 2, 4t, 2t)$*-difference set in* Q_{8t} *for every* t *of the form*

$$t = 2^a 10^b 26^c m$$

with $a, b, c \geq 0$.

Theorem 4.5 *Let* m *be a positive integer such that* $2m-1$ *and* $4m-1$ *is a prime power or* m *is odd and there is a Williamson matrix over* $\mathbb{Z}_m$*. Then there exists a* $\mathrm{BH}(2v, 4)$ *for all* $v = 2^a 10^b 26^c m$ *with* $a, b, c \geq 0$*. In particular there exists a* $\mathrm{BH}(2v, 4)$ *for all* $v \leq 46$.

Remark 4.6 To our knowledge, the only known $\mathrm{BH}(70,4)$ is due to Đokovic [10], via good matrices of order 35. We reconstructed this matrix, to compare with the matrix we constructed by our method from the element of $\mathrm{PUGP}(35,4,1)$ corresponding to the element of $\mathrm{PUGP}(70,2,1)$ listed in [16]. The two matrices are inequivalent.

5 Constructions

Many constructions for different Golay type sequences have been developed, see any of [2, 8, 11, 23] for examples. In this section we generalize known constructions. These are mostly recursive, and rely on sequences at small lengths that have been found by computational methods. We let $a \otimes b = [a_i b]$, and let $a \circ b$ denote the concatenation of a and b. Let k be even.

Proposition 5.1 *Let* $(a,b) \in \mathrm{PUGP}(v,k,m)$. *Let* $c = [a_0, b_0, a_1, b_1, \ldots, a_{v-1}, b_{v-1}]$ *and* $d = [a_0, -b_0, a_1, -b_1, \ldots, a_{v-1}, -b_{v-1}]$. *Then* $(c,d) \in \mathrm{PUGP}(2v,k,m)$.

Since Proposition 5.1 applies for all m, by Theorem 2.2 we may substitute $\mathrm{UGP}(v,k)$ for $\mathrm{PUGP}(v,k,m)$ in the proposition statement.

Corollary 5.2 *If* $\mathrm{UGP}(v,k) \neq \varnothing$ *then* $\mathrm{UGP}(2^n v,k) \neq \varnothing$ *for all* $n \geq 1$. *Likewise, if* $\mathrm{PUGP}(v,k,m) \neq \varnothing$ *then* $\mathrm{PUGP}(2^n v,k,m) \neq \varnothing$ *for all* $n \geq 1$.

Turyn's composition of Golay pairs was generalized in [7] to where one pair is in $\mathrm{CGP}(v)$, and we can extend this to where one pair is $\mathrm{UGP}(v,k)$ for any even k.

Proposition 5.3 *Let* $(a,b) \in \mathrm{GP}(g)$ *and* $(c,d) \in \mathrm{UGP}(v,k)$, *and define*

$$e = \frac{a+b}{2} \otimes c + \frac{a-b}{2} \otimes d^* \quad \textit{and} \quad f = \frac{a+b}{2} \otimes d - \frac{a-b}{2} \otimes c^*.$$

Then $(e,f) \in \mathrm{UGP}(gv,k)$.

Proof Let $s = qv + r$ where $r < v$ and $q < g$. Then $\mathrm{CAF}_s(e,f)$ is given by

$$\begin{aligned}
&\textstyle\sum_{j=0}^{g-q-1}\Big((a_j+b_j)(a_{j+q}+b_{j+q})\sum_{i=0}^{v-r-1} c_i\overline{c_{i+r}} + d_i\overline{d_{i+r}} \\
&\qquad +(a_j+b_j)(a_{j+q}-b_{j+q})\textstyle\sum_{i=0}^{v-r-1} c_i d_{-1-i-r} - d_i c_{-1-i-r} \\
&\qquad +(a_j-b_j)(a_{j+q}+b_{j+q})\textstyle\sum_{i=0}^{v-r-1} \overline{d_{-1-i}}\,\overline{c_{i+r}} - \overline{c_{-1-i}}\,\overline{d_{i+r}} \\
&\qquad +(a_j-b_j)(a_{j+q}-b_{j+q})\textstyle\sum_{i=0}^{v-r-1} \overline{d_{-1-i}}\,d_{-1-i-r} + \overline{c_{-1-i}}\,c_{-1-i-r}\Big) \\
&+\textstyle\sum_{j=0}^{g-q-2}\Big((a_j+b_j)(a_{j+q+1}+b_{j+q+1})\sum_{i=v-r}^{v-1} c_i\overline{c_{i+r}} + d_i\overline{d_{i+r}} \\
&\qquad +(a_j+b_j)(a_{j+q+1}-b_{j+q+1})\textstyle\sum_{i=v-r}^{v-1} c_i d_{-1-i-r} - d_i c_{-1-i-r} \\
&\qquad +(a_j-b_j)(a_{j+q+1}+b_{j+q+1})\textstyle\sum_{i=v-r}^{v-1} \overline{d_{-1-i}}\,\overline{c_{i+r}} - \overline{c_{-1-i}}\,\overline{d_{i+r}} \\
&\qquad +(a_j-b_j)(a_{j+q+1}-b_{j+q+1})\textstyle\sum_{i=v-r}^{v-1} \overline{d_{-1-i}}\,d_{-1-i-r} + \overline{c_{-1-i}}\,c_{-1-i-r}\Big).
\end{aligned}$$

Note that each of the sums over i above are zero, implying that the entire sum is zero.

We can also generalize to compositions of k_1-ary and k_2-ary sequences where $k_1 \neq k_2$. Let $k = \mathrm{lcm}(k_1, k_2)$.

Proposition 5.4 *Let* $(a, b) \in \mathrm{UGP}(v_1, k_1)$ *and* $(c, d) \in \mathrm{UGP}(v_2, k_2)$*, and let* $v = 2v_1v_2$*, and* $k = \mathrm{lcm}(k_1, k_2)$*. Then let* $e = (a \otimes c) \circ (b \otimes d^*)$ *and* $f = (a \otimes d) \circ (-b \otimes c^*)$*. Then* $(e, f) \in \mathrm{UGP}(v, k)$*.*

In [8] the construction of Proposition 5.4 is further improved for complex Golay pairs by extending it to a construction of an element of $\mathrm{CGP}(gv_1v_2)$ given any element of $\mathrm{GP}(g)$ for g even. This improvement may also be generalized to k-ary sequences, but first we need some further definitions. Let a and b be sequences of length v with entries in $\{0\} \cup \langle \zeta_k \rangle$. Then, as usual a and b are complementary if $\mathrm{CAF}_s(a, b) = 0$ for all $1 \leq s \leq v - 1$. Furthermore, we say a and b are *disjoint* if at least one of a_i and b_i are zero for all $0 \leq i \leq v - 1$. The *weight* of a disjoint pair (a, b) is the number of non-zero entries in $a + b = [a_i + b_i]$.

Theorem 5.5 *Let* a *and* b *be disjoint complementary* k_1*-ary sequences of length* g *and weight* g*, and let* $(c, d) \in \mathrm{UGP}(v, k_2)$*. Then let* $e = (c \otimes a) + (d^* \otimes b)$ *and* $f = (c^* \otimes b) - (d \otimes a)$*. Then* $(e, f) \in \mathrm{UGP}(gv, k)$ *where* $k = \mathrm{lcm}(k_1, k_2)$*.*

Proof By construction e and f are k-ary sequences. Let $s = qv + r$ where $r < v$ and $q < g$. Then $\mathrm{CAF}_s(e, f) = \mathrm{CAF}_q(c, d)\mathrm{CAF}_r(a, b) + \mathrm{CAF}_{q+1}(c, d)\mathrm{CAF}_{g-r}(\hat{a}, \hat{b}) = 0$.

Adhering to the notation of Theorem 5.5 but with $(c, d) \in \mathrm{PUGP}(v, k_2, m)$ for some m, it can similarly be shown that the sequences e and f satisfy $\mathrm{PPAF}_{m',s}(e, f) = 0$ for all $1 \leq s \leq gv - 1$, where $m' = m(k/k_2)$.

Corollary 5.6 *Let* a *and* b *be disjoint complementary* k_1*-ary sequences of length* g *and weight* g*, and let* $(c, d) \in \mathrm{PUGP}(v, k_2, m)$*. Then there exists* $(e, f) \in \mathrm{PUGP}(gv, k, m')$ *where* $k = \mathrm{lcm}(k_1, k_2)$ *and* $m' = m(k/k_2)$*.*

Now we are in a position to generalize [8, Theorem 5].

Proposition 5.7 *Let* g, k_1 *and* k_2 *be even. Suppose there is* $(a, b) \in \mathrm{GP}(g)$*,* $(c, d) \in \mathrm{UGP}(v_1, k_1)$ *and* $(e, f) \in \mathrm{UGP}(v_2, k_2)$*. Then there is* $(x, y) \in \mathrm{UGP}(v, k)$ *where* $v = gv_1v_2$ *and* $k = \mathrm{lcm}(k_1, k_2)$*.*

Proof Let $u = (a + a^* + b - b^*)/4$ and $v = (a - a^* + b + b^*)/4$. By [8, Lemma 4], u and v are disjoint complementary sequences of weight $g/2$. Further let $s = (c \otimes u) + (d \otimes v)$ and $t = (c \otimes v^*) - (d \otimes u^*)$, so s and t are disjoint and complimentary k_1-ary sequences of length and weight gv_1. Finally let $x = (e \otimes s) + (f^* \otimes t)$ and $y = (e^* \otimes t) - (f \otimes s)$. Then by Theorem 5.5, $(x, y) \in \mathrm{UGP}(v, k)$.

Corollary 5.8 *By Corollary 5.6 we can let* $(e, f) \in \mathrm{PUGP}(v_2, k_2, m)$ *in Proposition 5.7 and then derive that* $(x, y) \in \mathrm{PUGP}(v, k, m')$ *where* $m' = m(k/k_2)$*.*

Remark 5.9 It is verified in [8] that $\mathrm{UGP}(15, 4) = \varnothing$, and thus the existence of elements in $\mathrm{UGP}(v_1, 4)$ and $\mathrm{UGP}(v_2, 4)$ cannot guarantee the existence of a $\mathrm{UGP}(v_1v_2, 4)$.

Acknowledgements

This work has been fully supported by Croatian Science Foundation under the project 1637. The author would like to thank the reviewers for helpful suggestions that improved the quality of this paper.

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