Noname manuscript No.
(will be inserted by the editor)

On equivalence of negaperiodic Golay pairs

Ronan Egan

the date of receipt and acceptance should be inserted later

Abstract Associated pairs as defined by Ito [15] are pairs of binary sequence of length $2t$ satisfying certain autocorrelation properties that may be used to construct Hadamard matrices of order 4t. More recently, Balonin and Đoković [2] use the term negaperiodic Golay pairs. We define extended negaperiodic Golay pairs and prove a one-to-one correspondence with central relative $(4t, 2, 4t, 2t)$-difference sets in dicyclic groups of order $8t$. We present a new approach for computing negaperiodic Golay pairs up to equivalence, and determine conditions where equivalent pairs correspond to equivalent Hadamard matrices. We complete an enumeration of negaperiodic Golay pairs of length $2t$ for $1 \leq t \leq 10$, and sort them into equivalence classes. Some structural properties of negaperiodic Golay pairs are derived.

Keywords negaperiodic Golay pair, Hadamard matrix, relative difference set, binary sequences

Mathematics Subject Classification (2000) 05B10, 05B20, 11B99

1 Introduction

A *Hadamard matrix* is an $n \times n$ matrix H with entries in $\{\pm 1\}$ such that $HH^\top = I_n$ and is conjectured to exist for all n divisible by 4. Two $n \times n$ $\{\pm 1\}$-matrices $H_1 \approx H_2$ are *equivalent* if there exists $P, Q \in \mathrm{Mon}(n, \{\pm 1\})$ such that $PHQ^\top = H$. A $\{\pm 1\}$-matrix H is *cocyclic* over a group G if $H \approx [\psi(g,h)]_{g,h \in G}$ where $\psi : G \times G \to \{\pm 1\}$ is a 2-*cocycle*, i.e., satisfies $\psi(g,h)\psi(gh,k) = \psi(g,hk)\psi(h,k)$ for all $g,h,k \in G$. The notion of cocyclic development is much broader in general; see [7,14]. It is also conjectured that there exists a cocyclic Hadamard matrix of order $4m$ for all $m \geq 1$. Cocyclic development has proven to be a useful tool in the construction of Hadamard matrices and other pairwise combinatorial

Ronan Egan
Department of Mathematics, University of Rijeka
E-mail: ronan.egan@math.uniri.hr

designs; see [11, 16] for examples. The relationship between 2-cocycles of Hadamard matrices and relative difference sets is well known In [12] it is shown that a cocyclic Hadamard matrix of order $4t$ exists if there is a central relative $(4t, 2, 4t, 2t)$-difference set (CRDS) in a group of order $8t$ (see also [8, Theorem 2.4]). It is this relationship that strongly motivates this paper. Dicyclic groups of order $8t$, denoted Q_{8t} in this paper, have shown to be a rich source for CRDSs. Schmidt [17] proves the existence of a CRDS in Q_{8t} for $1 \leq t \leq 46$. This result derives from known orders of Golay pairs and Williamson matrices and a list of computed CRDSs in Q_{8t} for all $1 \leq t \leq 11$ in [12].

The *aperiodic autocorrelation function* of $\{\pm 1\}$-sequence a of length v and shift k is defined to be

$$\mathrm{AF}_k(a) = \sum_{i=0}^{v-k-1} a_i a_{i+k}.$$

A *Golay pair* [13] is a pair of sequences (a, b) of length v such that $\mathrm{AF}_k(a) + \mathrm{AF}_k(b) = 0$ for all $1 \leq k \leq v-1$. We denote the set of all Golay pairs of length v by GP_v. In [2] the authors define the *negaperiodic autocorrelation function* of sequence a and shift k to be $\mathrm{NAF}_k(a) = a \cdot aN^k$ where N is the negashift matrix given by

$$N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & & 0 & 1 \\ -1 & 0 & 0 & & 0 & 0 \end{bmatrix}. \tag{1}$$

This leads to the concept of a *negaperiodic Golay pair* (NGP). These are pairs of $\{\pm 1\}$ sequences (a, b) of even length v such that

$$\mathrm{NAF}_k(a) + \mathrm{NAF}_k(b) = 0 \text{ for all } 1 \leq k \leq v-1. \tag{2}$$

Any Golay pair is also a NGP. We let NGP_v denote the set of NGPs of length v. A NGP_{2t} can be used to directly construct a Hadamard matrix of order $4t$. NGPs are precisely what Ito refers to as *associated pairs* in [15]. Therein associated pairs of order $2t$ are listed for all $1 \leq t \leq 45$. Due to this, if (a, b) is a NGP, we say b is an *associate* of a and vice versa. The existence of a NGP_{2t} is precisely the condition given in [12, Section 6] for the existence of a certain cocyclic Hadamard matrix over $\mathrm{D}_{4t} = \langle x, y \ : \ x^2t = y^2 = 1, yx = x^{-1}y \rangle$, the dihedral group of order $4t$. In this case the matrix is cocyclic over an element in the cocycle class $[\psi]$ where the representative 2-cocycle ψ is given by

$$\psi(x^i, x^j y^k) = \begin{cases} 1 & i + j < 2t, \\ -1 & i + j \geq 2t, \end{cases}$$

$$\psi(x^i y, x^j y^k) = \begin{cases} (-1)^k & i \geq j, \\ (-1)^{k+1} & i < j, \end{cases}$$

for $0 \leq i, j < 2t$ and $0 \leq k \leq 1$.

NGPs correspond exactly to objects that were referred to as *suitable pairs* in the authors PhD thesis [10]. In this paper we will use the term extended negaperiodic Golay pair in light of the appearance of [2].

This paper is outlined as follows. Section 2 contains the relevant background and necessary definitions for the remainder of the paper. We introduce the extended NGPs and outline a one-to-one correspondence between the set of NGPs of length $2t$, extended NGPs, and CRDSs in Q_{8t}. We discuss equivalence operations on NGPs in Section 3, and construct matrix groups that act on $\{\pm 1\}$-vectors such that the orbits correspond to equivalence classes, preserving the property of being a NGP. We also prove that in most but not all cases, if two NGPs are equivalent then so are the corresponding Hadamard matrices. In Section 4 we outline some properties of a NGP_{2t}. We present some computational results in Section 5; in particular we use MAGMA [3] programs to enumerate NGP_{2t} for $1 \leq t \leq 10$, and sort them into equivalence classes.

2 Preliminaries

Throughout this paper $\mathrm{Q}_{8t} = \langle x, y \mid x^{4t} = 1, y^2 = x^{2t}, x^y = x^{-1} \rangle$ is the dicyclic group of order $8t$; so Q_{8t} has element set $\{x^i y^j \mid 0 \leq i \leq 4t-1, 0 \leq j \leq 1\}$, and central subgroup $Z = \{1, x^{2t}\}$. I_n will denote the identity matrix of order n, and we write I when the order is clear from context. Let N be the negashift matrix as defined by (1). Binary sequences are $\{\pm 1\}$-seqeunces and we write $-$ for -1 in sequences for clarity. Sequences are indexed beginning at 0, and indices should be read modulo the length of the sequence.

2.1 Central relative difference sets

Let E be a group of order vm with a normal subgroup Z of order m. Suppose that R is a k-subset of E, such that the multiset of quotients $r_1 r_2^{-1}$, $r_i \in R$, $r_1 \neq r_2$, contains each element of $E \setminus Z$ exactly λ times, and contains no element of Z. Then R is called a (v, m, k, λ)-*relative difference set* in E with *forbidden subgroup* Z. If Z is a central subgroup of E then we call R a *central relative difference set*.

For certain parameters, the existence of relative difference sets is equivalent to the existence of cocyclic pairwise combinatorial designs such as Hadamard matrices; see [8] and [7, Chapters 10, 15]. The following motivates this paper.

Theorem 1 *There is a cocyclic Hadamard matrix over a group G of order $4t$ if and only if there is a $(4t, 2, 4t, 2t)$-central relative difference set in an extension E of $\langle -1 \rangle$ by G.*

The group E in Theorem 1 is an extension group of the cocyclic Hadamard matrix, also called a *Hadamard group*. For the remainder of this paper the default parameters for a CRDS in Q_{8t} are $(4t, 2, 4t, 2t)$.

Let a be a $\{\pm 1\}$ sequence of length n. As in [1], we define the *periodic autocorrelation* of a $\{\pm 1\}$-sequence a of period (or length) n with shift k to be

$$C_k(a) = \sum_{i=0}^{n-1} r_i r_{i+k}$$

where subscripts are taken modulo n. We note the following.

Lemma 2 *Let a be a sequence of length $2t$ and let $a \circ a'$ be the concatenation of a and $-a$. Then $C_k(a') = 2\mathrm{NAF}_k(a)$ for all k.*

Proof This is a routine calculation. □

Adhering to Lemma 2, if $(a, b) \in \mathrm{NGP}_{2t}$, then a' and b' satisfy

$$C_k(a') + C_k(b') = 0 \tag{3}$$

for all $1 \leq k \leq 4t - 1$, $k \neq 2t$. We will refer to (a', b') as an *extended* NGP.

Theorem 3 *(a', b') is an extended NGP if and only if $R = \{x^i \mid a'_i = 1\} \cup \{x^i y \mid b'_i = 1\}$ is a CRDS in Q_{8t}.*

Proof Use the relations of Q_{8t} and equation (3) to prove in both directions. □

2.2 Constructions

Constructions of CRDSs in Q_{8t} are described in [17]. We outline here some constructions of NGPs. Let $(a, b) \in \mathrm{NGP}_{2t}$. It is readily checked that $c = [a_0, b_0, a_1, b_1, \ldots, a_{2t-1}, b_{2t-1}]$ and $d = [a_0, -b_0, a_1, -b_1, \ldots, a_{2t-1}, -b_{2t-1}]$ satisfy (2) for all $1 \leq k \leq 4t - 1$, i.e., $(c, d) \in \mathrm{NGP}_{4t}$.

Theorem 4 *If $\mathrm{NGP}_{2t} \neq \phi$, then $\mathrm{NGP}_{2^m t} \neq \phi$ for all $m \geq 1$.*

Remark 5 Given $(c, d) \in \mathrm{NGP}_{2v}$ such that $c_i = (-1)^i d_i$ for all i, then $(a, b) \in \mathrm{NGP}_v$ where $a = [c_{2i}]_{0 \leq i \leq v-1}$ and $b = [c_{2i+1}]_{0 \leq i \leq v-1}$.

Turyn [18] proved that Golay pairs exist at all lengths $l = 2^a 10^b 26^c$ where a, b, c are non-negative integers; they are not currently known to exist at other lengths. Golay sequences have been used to construct Hadamard matrices and other pairwise combinatorial designs; see, e.g., [4–6]. In [9, Section 4] two methods of constructing periodic Golay pairs of length vg from a periodic Golay pair of length $v = 2t$ and Golay pair of length g are outlined. These constructions also apply to NGPs, by replacing the periodic Golay pair with a NGP. We outline one method here [9, cf. Proposition 4.1].

Proposition 6 *Suppose that $(a,b) \in \mathrm{GP}_g$ and $(c,d) \in \mathrm{NGP}_{2t}$. For indeterminates u and v, let X be the sequence of length $2t$ given by*

$$X_i = \begin{cases} u, & \text{if } c_i = d_i = 1, \\ -u, & \text{if } c_i = d_i = -1, \\ v, & \text{if } c_i = 1,\ d_i = -1, \\ -v, & \text{if } c_i = -1,\ d_i = 1. \end{cases}$$

Then let Y be obtained from X by reversing X and simultaneously replacing u with v and v with $-u$. Finally, if (e,f) is obtained from X and Y by letting $u = a$ and $v = b$, $(e,f) \in \mathrm{NGP}_{2tg}$.

The second construction is Turyn's multiplication of Golay pairs [18] where one Golay pair is replaced by a NGP_{2t}.

In [2, Sections 5-7] two Paley type constructions are described. In the first case Paley conference matrices of order $q+1$ are used to construct a NGP_{q+1} for any odd prime power q. In the second case via similar means, a $\mathrm{NGP}_{(q+1)/2}$ is constructed for every odd prime power $q \equiv 3 \mod 4$. Pairs constructed with these methods belong to the first and second Paley series' respectively, and examples for all relevant values of q are listed in the appendices of [2] up to length $2t = 122$ for the first series, and $2t = 64$ in the second.

3 Equivalence of negaperiodic Golay pairs and Hadamard matrices

In [2] the authors list five elementary equivalence operations on pairs NGPs. As we will refer to these operations often, we introduce shorthand notation in the form of a function acting on pairs of sequences for each operation. The equivalence operations on a pair (a,b) are as follows:

(i) Reverse a or b; denote by $\pi_a^{(\mathrm{i})}$ or $\pi_b^{(\mathrm{i})}$.
(ii) (Shifting) Replace a or b with aN^s or bN^s respectively, for some s; denote by $\pi_{a,s}^{(\mathrm{ii})}$ or $\pi_{b,s}^{(\mathrm{ii})}$.
(iii) Switch a and b; denote by $\pi^{(\mathrm{iii})}$.
(iv) For k coprime to $2t$, replace both a and b with $(z_i a_{ki})_{i=0}^{2t-1}$ and $(z_i b_{ki})_{i=0}^{2t-1}$ respectively, where $z_i = 1$ if $(ki \mod 4t) < 2t$ and $z_i = -1$ otherwise; denote by $\pi_k^{(\mathrm{iv})}$.
(v) Negate every odd index entry in both a and b; denote by $\pi^{(\mathrm{v})}$.

For example, if (c,d) is the result of applying operation (i) to a followed by operation (ii) to b with $s = 3$, we write $(c,d) = \pi_{b,3}^{(\mathrm{ii})} \cdot \pi_a^{(\mathrm{i})}(a,b)$. Letting B denote the set of all binary sequences of length $2t$, we denote by G the group acting on $B \times B$ generated by these functions. Pairs of binary sequences in the same orbit of G are equivalent. Note that $\pi^{(\mathrm{v})} = \pi_b^{(\mathrm{i})} \cdot \pi_a^{(\mathrm{i})} \cdot \pi_{b,2t-1}^{(\mathrm{ii})} \cdot \pi_{a,2t-1}^{(\mathrm{ii})} \cdot \pi_{2t-1}^{(\mathrm{iv})}$, and is not required to generate G. Thus $|G| = 128t^2\varphi(2t)$ where φ is the Euler-phi function.

3.1 Matrix groups

In this section we construct a matrix representation $M < \mathrm{Mon}(4t, \langle -1 \rangle)$ of G which acts on $\{\pm 1\}$-vectors of length $4t$, i.e., concatenated pairs of sequences of length $2t$. Denote the vector of a concatenated pair of sequences (a,b) by $a \circ b$. Matrices act on the right of row vectors, and the orbits correspond to equivalence classes of $B \times B$, where either all pairs are in NGP_{2t}, or none are.

Let $\delta_x^y = 1$ if $x = y$ and 0 otherwise, and let $f(n)$ be the remainder after division of n by $2t$. We define $K_{(k)} = [\delta_i^{1+f((j-1)k)}(-1)^{z_{j-1}}]_{1 \leq i,j \leq 2t}$ where k is coprime to $2t$, and z is the sequence defined in equivalence operation (iv). Let $T = [\delta_i^j(-1)^{i-1}]_{1 \leq i,j \leq 2t}$, and let $R = [\delta_i^{2t+1-j}]_{1 \leq i,j \leq 2t}$. If (c,d) is the result of applying an elementary equivalence operation to (a,b) then we get $c \circ d$ by multiplying $a \circ b$ on the right by the following matrices:

(i) $\pi_a^{(\mathrm{i})}$ or $\pi_b^{(\mathrm{i})}$: $\begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}$ or $\begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix}$,

(ii) $\pi_{a,s}^{(\mathrm{ii})}$ or $\pi_{b,s}^{(\mathrm{ii})}$: $\begin{bmatrix} N^s & 0 \\ 0 & I \end{bmatrix}$ or $\begin{bmatrix} I & 0 \\ 0 & N^s \end{bmatrix}$,

(iii) $\pi^{(\mathrm{iii})}$: $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$,

(iv) $\pi_k^{(\mathrm{iv})}$: $\begin{bmatrix} K_{(k)} & 0 \\ 0 & K_{(k)} \end{bmatrix}$,

(v) $\pi^{(\mathrm{v})}$: $\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}$.

Let M be the matrix group generated by these matrices. The orbits under M correspond to equivalence classes in $B \times B$, which are then tested for satisfying (2). Computational results are listed in Section 5. We may also test equivalence of known NGPs.

Example 7 *In [2] the authors question whether NGPs in the first and second Paley series can be equivalent, when they exist at the same length. Referring to the Paley series NGPs listed in the appendices of [2], we test the relevant NGPs for equivalence. We find that all pairs in* NGP_6 *are equivalent, and thus the question is answered positively in this instance. The pairs listed belonging to each series in* NGP_{10}, NGP_{14}, NGP_{30} *and* NGP_{42} *are inequivalent.*

3.2 Equivalent Hadamard matrices

A pair $(a,b) \in \mathrm{NGP}_{2t}$ can be used to construct a Hadamard matrix as follows. Let A and B be the nega-circulant matrices with first row a and b respectively. That is, row $A_{i+1} = A_i N$ and $B_{i+1} = B_i N$. Then the matrix $H = \begin{bmatrix} A & B \\ -B^\top & A^\top \end{bmatrix}$ is Hadamard. In this section we prove that, if we exclude the equivalence operation of reversing one but not both sequences, two Hadamard matrices corresponding to equivalent NGPs are also Hadamard equivalent. That is, for two matrices $H = \begin{bmatrix} A & B \\ -B^\top & A^\top \end{bmatrix}$ and $\overline{H} = \begin{bmatrix} \overline{A} & \overline{B} \\ -\overline{B}^\top & \overline{A}^\top \end{bmatrix}$ corresponding to pairs (a,b) and $(\bar{a},\bar{b})$ that are equivalent under operations (ii) - (v), we find signed permutation matrices (P,Q) such that $PHQ^\top = \bar{H}$. They correspond to the equivalence operations as follows.

- $\pi^{(\mathrm{ii})}_{a,s}$: $(P,Q) = \left(\begin{bmatrix} N^s & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & N^s \end{bmatrix} \right)$
- $\pi^{(\mathrm{ii})}_{a,s}$: $(P,Q) = \left(\begin{bmatrix} I & 0 \\ 0 & (N^s)^\top \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & (N^s)^\top \end{bmatrix} \right)$.
- $\pi^{(\mathrm{iii})}$: $(P,Q) = \left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right)$.
- $\pi^{(\mathrm{iv})}_{k}$: $(P,Q) = \left(\begin{bmatrix} K^\top_{(k)} & 0 \\ 0 & K^\top_{(k)} \end{bmatrix}, \begin{bmatrix} K^\top_{(k)} & 0 \\ 0 & K^\top_{(k)} \end{bmatrix} \right)$.
- $\pi^{(\mathrm{v})}$: $(P,Q) = \left(\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}, \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \right)$.

Note that $\pi^{(\mathrm{i})}_a \cdot \pi^{(\mathrm{i})}_b = \pi^{(\mathrm{v})} \cdot \pi^{(\mathrm{ii})}_{b,4t-1} \cdot \pi^{(\mathrm{ii})}_{a,4t-1} \cdot \pi^{(\mathrm{iv})}_{2t-1}$. Thus the Hadamard matrix obtained from (a,b) is equivalent to the one obtained from $\pi^{(\mathrm{i})}_b \cdot \pi^{(\mathrm{i})}_a(a,b)$. However it is not necessarily the case that reversing exactly one sequence yields an equivalent Hadamard matrix. As an example, let $a = [1,1,-,1,1,1,1,1]$ and $b = [1,1,-,1,-,1,1,1]$. The order 16 Hadamard matrices constructed from the pairs (a,b) and $\pi^{(\mathrm{i})}_b(a,b)$ are not equivalent. It is also possible that inequivalent pairs yield equivalent Hadamard matrices. The smallest length such that inequivalent pairs in NGP_{2t} exist is 10. The inequivalent pairs

$$([1,1,-,1,-,-,-,-,1,1],[1,-,1,-,-,-,1,-,-,-])$$

and

$$([1,-,1,-,1,1,-,-,-,-],[-,1,1,1,-,-,-,1,-,-])$$

yield equivalent Hadamard matrices.

4 Structure of negaperiodic Golay pairs

In general a very small proportion of sequences are suitable candidates for NGPs. In order to reduce computing time, we want to limit search spaces by studying properties of NGPs. Ito [15] derives several necessary conditions regarding runs (blocks) in the pair of sequences. A *run* in a sequence a is a subsequence of a where all elements have the same value. Binary sequences can be completely described by their run structure. For example, the sequence $[1,1,-,-,-,1]$ may be written $[2,3,1]$ with the understanding that the first entry in the sequence is 1. Since it is always possible to replace a sequence a of length v with $\bar{a} = aN^s$ for some s such that $\bar{a}_0 = \bar{a}_{v-1} = 1$, we will usually assume that sequences begin and end with a 1, and thus have an odd number of runs. The following are samples of results derived from [15, cf. Propositions 4,5,6].

Lemma 8 *The number of runs in the sequences of* $(a,b) \in \mathrm{NGP}_{2t}$ *must sum to* $2t$.

Proof Deny; then $\mathrm{NAF}_1(a) + \mathrm{NAF}_1(b) \neq 0$. □

Lemma 9 *The number of runs in either sequence a or b is bounded below by $\lfloor \frac{t+1}{2} \rfloor$.*

Proof Suppose there are m runs in a. By Lemma 8 there are $2t - m$ runs in b. This implies that $\mathrm{NAF}_2(a) \geq 2t - 4m$ and $\mathrm{NAF}_2(b) \geq 2t - 4m$ and thus $\mathrm{NAF}_2(a) + \mathrm{NAF}_2(b) \geq 4t - 8m$. Therefore $8m \geq 4t$, proving the result. □

The following derives from [10, Lemma 8.3.19].

Lemma 10 *Let $(a, b) \in \mathrm{NGP}_{2t}$, $t > 1$, where each sequence begins and ends with 1. There are exactly t runs of length 1 in the sequences.*

Proof Suppose that there are u runs of length 1. Let v be the number of runs of length greater than 2 in the sequences combined and let l_i be the length of each of these for $1 \leq i \leq v$. By Lemma 8 the average length of a run in the sequences a and b combined is 2, and thus $\sum_{i=1}^{v}(l_i - 2) = u$. This implies that u terms of $\mathrm{NAF}_2(a) + \mathrm{NAF}_2(b)$ are products of entries within the same runs and must equal 1. If a run of length 1 in a or b in position $j \notin \{0, 2t-1\}$, then $a_{j-1} = a_{j+1}$ or $b_{j-1} = b_{j+1}$, and if $j \in \{0, 2t-1\}$, then $a_{j-1} = -a_{j+1}$ or $b_{j-1} = -b_{j+1}$. This implies that a further u terms of $\mathrm{NAF}_2(a) + \mathrm{NAF}_2(b)$ are 1. All other terms are -1, and hence $u = t$. □

Corollary 11 *The maximum length of a run in a or b is $l = t + 2$.*

Several similar restrictions are proven in [15]. In our computations outlined in the next section, we find that for $t \leq 10$ there exists at least one equivalence class not of maximal order. The construction outlined prior to Theorem 4 ensures the existence of $(a, b) \in \mathrm{NGP}_{4t}$ such that $\pi^{(\mathrm{v})} \cdot \pi^{(\mathrm{iii})}(a, b) = (a, b)$. However, by Remark 5, no such pair exists in NGP_{2t} for odd t. We also have the following.

Proposition 12 *If $(a, b) \in \mathrm{NGP}_{2t}$ then $a \neq aN^s$ for $1 \leq s \leq 2t - 1$.*

Proof Suppose $a = aN^s$. Then $\mathrm{NAF}_s(a) = 2t$ and thus $\mathrm{NAF}_s(b) = -2t$. But this would imply that $\mathrm{NAF}_{2s}(a) = \mathrm{NAF}_{2s}(b) = 2t$, contradicting (2) for all $s \neq t$ in the range stated. If $s = t$ then $a_0 = a_t = -a_0$, also a contradiction. □

For odd t, we find that equivalence classes not of maximal size are most often a consequence of a pair being fixed by operations that include reversing a sequence. It is pertinent therefore to consider sequences with symmetric properties. Let a be a sequence of length $2t$. Then a is *symmetric* if $a_i = a_{2t-1-i}$ for all $0 \leq i \leq t - 1$; a is *skew-symmetric* if $a_i = -a_{2t-1-i}$ for all $0 \leq i \leq t - 1$; and a is *quasi-symmetric* if $a_i = a_{2t-i}$ for all $1 \leq i \leq t - 1$. Let $(a, b) \in \mathrm{NGP}_{2t}$. If a is symmetric then $(a, b) = \pi_a^{(\mathrm{i})}(a, b)$, and if a is skew symmetric then $(a, b) = \pi_{a,t}^{(\mathrm{ii})} \cdot \pi_a^{(\mathrm{i})} \cdot \pi_{a,t}^{(\mathrm{ii})}(a, b)$. If a NGP belongs to the second Paley series, then one sequence is skew-symmetric, and its associate is quasi-symmetric. We consider the possible values $\mathrm{NAF}_k(a)$ can take if a has a type of symmetry.

Lemma 13 *Let a be a symmetric, skew-symmetric or quasi-symmetric sequence of length $2t$. Then*

$$\mathrm{NAF}_k(a) \equiv \begin{cases} 0 \mod 4, & \text{if } t \text{ and } k \text{ are either both odd or both even,} \\ 2 \mod 4, & \text{otherwise.} \end{cases}$$

Proof Let a be symmetric. If k is even, t terms appear twice in the sum $a \cdot aN^k$, whereas if k is odd, two terms cancel each other out and $t-1$ other terms appear twice. The result follows. If a is skew-symmetric then aN^t is symmetric, and $\mathrm{NAF}_k(a) = \mathrm{NAF}_k(aN^t)$ for all k, so we get the same result. The proof for quasi-symmetric sequences is similar. □

Lemma 13 implies that if a is a symmetric or quasi-symmetric sequence then its associate b also has the property of Lemma 13, and thus another symmetric or quasi-symmetric sequence is likely. This will not necessarily be the case as the pairs in NGP_{2t}, for $t \in \{7, 9\}$ listed in Table 1 with one symmetric sequence indicate.

$t=7$	[3,2,1,2,1,2,3]	[2,1,1,1,1,1,7]
$t=9$	[1,1,1,2,8,2,1,1,1]	[3,2,4,1,1,2,2,1,2]

Table 1: NGP_{2t} with one symmetric sequence

5 Computational results

Equivalence classes of NGPs can be of any order dividing $128t^2\varphi(2t)$. Table 2 gives the number $n(t)$ of pairs in NGP_{2t}, and the number $d(t)$ gives the number of equivalence classes in each case.

t	1	2	3	4	5	6	7	8	9	10
$n(t)$	16	128	576	4096	11200	59904	90944	557056	1041984	4172800
$d(t)$	1	1	1	1	3	8	5	13	20	59

Table 2: Equivalence classes of negaperiodic Golay pairs

The size of these equivalence classes in the $d(t)$ row of Table 2 are listed below.

- $t = 1$: 1 class of size 16.
- $t = 2$: 1 class of size 128.
- $t = 3$: 1 class of size 576.

- $t = 4$: 1 class of size 4096.
- $t = 5$: 1 class of size 6400; 1 class of size 3200; 1 class of size 1600.
- $t = 6$: 5 classes of size 9216; 3 classes of size 4608.
- $t = 7$: 1 class of size 37632; 2 classes of size 18816; 1 class of size 9408; 1 class of size 6272.
- $t = 8$: 4 classes of size 65536; 9 classes of size 32768.
- $t = 9$: 14 classes of size 62208; 5 classes of size 31104; 1 class of size 15552.
- $t = 10$: 26 classes of size 102400; 27 classes of size 51200; 4 classes of size 25600; 2 classes of size 12800.

The second Paley series accounts for at least one class not of maximal size found for $t \in \{1, 2, 3, 5, 6, 7, 8\}$. Theorem 4 accounts for at least one class not of maximal size for all even t. The time required for complete classifications of NGP_{2t} increases exponentially with t, but testing equivalence is a very quick process. All computations were carried out in MAGMA [3].

Acknowledgements

This work has been supported in part by Croatian Science Foundation under the project 1637. Parts of this work appeared in the author's Ph.D. thesis written under the supervision of Professor Dane Flannery and was supported by the Irish Research Council (Government of Ireland Postgraduate Scholarship) and National University of Ireland, Galway (Hardiman Fellowship).

References

1. K. T. Arasu, Sequences and arrays with desirable correlation properties, Information security, coding theory and related combinatorics, 136–171, NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur., 29, IOS, Amsterdam, (2011).
2. N. A. Balonin, D. Ž. Đoković, Negaperiodic Golay pairs and Hadamard matrices, http://arxiv.org/abs/1508.00640, (2015).
3. W. Bosma, J. Cannon, C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., 24, 235–265, (1997).
4. G. Cohen, D. Rubie, J. Seberry, C. Koukouvinos, S. Kounias, M. Yamada, A survey of base sequences, disjoint complementary sequences and $\mathrm{OD}(4t; t, t, t, t)$, J. Combin. Math. Combin. Comput., 5, 69–104, (1989).
5. R. Craigen, Signed groups, sequences, and the asymptotic existence of Hadamard matrices, Journal of Combin. Theory Ser. A, 71, 241–254, (1995).
6. R. Craigen, W. Holzmann, H. Kharaghani, Complex Golay sequences: structure and applications, Discrete Math., 252, 73–89, (2002).
7. W. de Launey, D. L. Flannery, *Algebraic design theory*, Math. Surveys Monogr., 175, American Mathematical Society, Providence, RI, (2011).
8. W. de Launey, D. L. Flannery, K. J. Horadam, Cocyclic Hadamard matrices and difference sets, Discrete Appl. Math., 102, 47–61, (2000).

9. D. Ž. Doković and I. S. Kotsireas, Periodic Golay pairs of length 72, *Algebraic Design Theory and Hadamard Matrices*, Springer Proceedings in Mathematics & Statistics 133, 83–92, (2015).
10. R. Egan, *Topics in cocyclic development of pairwise combinatorial designs*, PhD thesis, National University of Ireland, Galway (2015).
11. R. Egan, D. L. Flannery, and P. Ó Catháin, Classifying cocyclic Butson Hadamard matrices, *Algebraic Design Theory and Hadamard Matrices*, Springer Proceedings in Mathematics & Statistics 133, 93–106, (2015).
12. D. L. Flannery, Cocyclic Hadamard matrices and Hadamard groups are equivalent, J. Algebra, 192, 749–779, (1997).
13. M. J. E. Golay, Multislit spectroscopy. J. Opt. Soc. Am. 39: 437–444, (1949).
14. K. J. Horadam, *Hadamard matrices and their applications*, Princeton University Press, Princeton, NJ, (2007).
15. N. Ito, On Hadamard groups IV, J. Algebra, 234, 651–663, (2000).
16. P. Ó Catháin, M. Röder, The cocyclic Hadamard matrices of order less than 40, Des. Codes Cryptogr., 58, no. 1, 73–88, (2011).
17. B. Schmidt, Williamson matrices and a conjecture of Ito's. Des. Codes Cryptogr., 17, 61–68, (1999).
18. R. J. Turyn, Hadamard matrices, Baumert-Hall units, four-symbol sequences, pulse compression, and surface wave encodings, J. Combin. Theory (A), 313–333, (1974).