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Proceedings from ICME 15, Topic Study Group 1.2: Teaching and Learning of Early Algebra

Aisling Twohill · Isil Isler-Baykal · Jodie Miller · Eric Knuth · Alessandro Ribeiro *Editors*

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PREFACE

Within these proceedings we present the papers submitted to the Early Algebra Topic Study Group (TSG 1.2) of the 15th International Congress of Mathematics Education (ICME) held in Sydney, Australia in July 2024.

The papers presented at TSG 1.2 of ICME 15 represented a deep and broad understanding of early algebra and many interesting and novel findings were shared that prompted robust discussions. Established themes of early algebra, such as the equal sign and functional thinking were the focus of presentations, along with relatively less studied areas related to early algebra, such as cultural relevance and sustainability.

In this preface we will summarise key themes and highlight trends in ideas and in research foci. We invite the reader, however, to access the individual papers for more detailed and specific discussions and findings relating to various elements of Early Algebra.

Key themes

Multiple representations

Throughout the TSG sessions, the role of representations was foregrounded, including the following:

Culturally relevant representations: A significant proportion of the research represented at ICME 15 TSG 1.2 highlighted the critical role of representations in children's developing algebraic thinking. Gibbs' research in New Zealand emphasises the inclusion of cultural relevance as a design focus of tasks for early algebra in ensuring that children feel connected to the content and empowered to succeed.

Visualisation: Many studies highlighted the role played by visual representations. Moreno, for example, described how students used graphs as tools to think with when they reasoned about a functional relationship drawn from the real-world context of a fairground.

Using digital tools: Panorkou and Provost demonstrated how participants in their study succeeded in reasoning multivariationally when manipulating digital representations of three simulations.

Concrete: Wilkie and Hopkins' study demonstrated how physical representations, vertical towers of blocks in this case, can support a robust relational understanding of equivalence that facilitates compensation approaches to subtraction. Ji's study also aimed to explore the role of physical movement, using the concrete and digital pan balance, in children's understanding of equivalence.

Real-world: Real-world representations are shown in the research of Moreno and Adamuz-Povedano et al. to support children in navigating the structure of functional relationships.

Besides a focus on multiple representations, the contributions also touched upon several areas related to early algebra. These include the equal sign and equations, generalized arithmetic thinking, teacher education, communication of thinking, sustainability, and embodied cognition.

The equal sign and equations

Sun et al. focused on Chinese grade 5 students' understanding of equal sign and found that the majority of the students showed a relational understanding of the equal sign. They hypothesized that exposure to simple nonconventional forms of equation (e.g., $5 = x - 2$) and balance models might have helped students with the relational understanding of the equal sign. Similarly, Ji focused on how Chinese students conceptualized the equal sign and equations in a classroom activity using the pan-balance model. Ohta's study included a focus on developing an assessment task about part-whole relationships for lower elementary grades in Japan.

Generalized arithmetic

Wilkie and Hopkins' study focused on 9-11-year-old students', Timothy's in this paper, relational thinking about the compensation property of equality with subtraction tasks, using vertical towers of blocks.

Functional thinking

Similar to generalised arithmetic, research consistently highlighted the challenges of abstraction, and typical errors or misconceptions that stymied children's functional thinking. Balancing such research, Lourdes Anglada et al. described very young children's successful functional thinking as they move between physical and iconic representations using tables. Gibbs focused on 10 to 12 years old Māori and Pacific students' development of functional thinking using culturally located tasks. Panorkou and Provost focused on sixth-grade students' multivariational reasoning using digital simulations. Moreno described 10-11 years old students' generalization of functional relationships using graphs of contextualized tasks. Adamuz-Povedano et al. similarly focused on 9-10 years old students' generalization of functional relationships in interactive groups.

Teacher education

Pinto et al. presented research from their work with pre-service teachers in Chile using the framework of professional noticing of children's mathematical thinking, where they acknowledged the need for more focus on teacher education. For example, their research highlighted the role played by pre-service teachers' understandings in their decisions about how to respond to children's thinking and is thus well placed to inform teacher education in the domain of early algebra.

Communication of thinking

Araya's research, focusing on the case study of Daniel, a third-grade Haitian student, and a second-language learner, raised very important questions about how we assess children's development of algebraic thinking, specifically functional thinking, and the implications of children's capacity to communicate in the language of the classroom.

Sustainability

Fred et al. explored the connection between early algebra and sustainability issues through dilemmas that were aimed to be naturally embedded in algebraic wicked problems (AWPs). A team of pre-and in-service teachers and researchers met over ten sessions and developed such problems. Drawing from observations of enactment and group meetings centred around Algebraic Word Problems (AWPs), the researchers posited that dilemmas, coupled with the teacher's questions, were pivotal in enhancing algebraic thinking and integrating environmental topics in students' work. This suggests a promising approach to intertwining sustainability aspects with early algebra education by employing dilemmas in AWPs.

Embodied cognition

Ji's study aims to explore the role of physical movement in children's understanding of equivalence. Araya's study focused on gesture as a form of communication, but also demonstrates a child, Daniel, enacting the structure of the pattern through his gestures.

Areas for further investigation in future TSGs in this area:

The overarching recommendation from presenters and attendees was for the need for more research on teachers and teacher education in the area of early algebra. Research presented at TSG 1.2 of ICME 15 was dominated by a focus on children's thinking. All attendees acknowledged the centrality of such research but emphasised the need for evidence-informed interventions that could result in high-quality teacher in-service and pre-service education in early algebra.

Over the final two days of ICME 15, the co-chairs gathered feedback and talking points from the attendees through the use of a padlet. Through the padlet, one attendee also emphasised the focus on cultural context and diversity as "... it is necessary to develop research beyond the typical mathematics classroom; the importance of cultural contexts and diversities that make the development of algebraic thinking in the algebra class a fairer and more equitable space."

CONCLUSION

Having engaged with the papers presented in this proceeding, and the presentations made at ICME 15, we see great merit in drawing attention to the role of representations in communicating algebraic thinking. In referring to communicating, we are referring to both the representations used by the teacher to allow children to explore structure through resources that mediate the abstract and facilitate dynamic interactions, and also the means of communication that are available to the children. Central to investigation of these representations is the role of the cultural context of the children and the maximisation of accessibility and removal of barriers – are these contexts 'real' from the children's perspective, do they resonate, do they make the children feel that this mathematics is for them or of relevance to their community? An intrinsic priority of early algebra is in downplaying the role of algebra as 'gatekeeper' and attention should therefore be paid on how representations may maximise accessibility.

Also, the contributions were from several areas related to early algebra including equals sign and equations, generalized arithmetic, functional thinking, teacher education, communication of thinking, sustainability, and embodied cognition from various countries and cultures. This shows the promise of early algebra studies to help students and teachers of various backgrounds to develop algebraic thinking using multiple tools and in relation to many aspects.

CONTENTS

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FIVE-YEAR-OLDS EXPRESSING RELATIONSHIPS WHILE WORKING WITH A FUNCTION MACHINE: APPROACHES TO THE USE OF FUNCTION TABLES

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This study describes the work with 5-years-old children on a problem involving a function machine. Our objectives were: (a) to describe the written representations used by children to organize the relationships between covarying quantities; and (b) to compare the characteristics of these representations with conventional function tables. We focused on the third session of a Classroom Teaching Experiment (CTE) as that is where children represented the perceived relationship using an open format. Broadly speaking, results show that, whereas most of the children represented the relationships pictorially, they did not connect them with the elements involved in the function machine. However, seven children showed evidence of relationships between variables with different levels of sophistication. Some of them used tables with specific characteristics, which are detailed below. We conclude with some of the implications of introducing function tables at this age as a way to support algebraic thinking.

In spite of the recognized importance of algebraic thinking on the international research agenda, there are more studies focusing on primary education (ages 6-12) and secondary education (ages 13-16) (Sibgatullin et al., 2022). The few studies which address algebraic topics in preschool (ages 3-6) look primarily at the work with patterns (Pincheira et al., 2022). Therefore, it is necessary to find out how children interact with different contents that promote algebraic thinking at these ages.

This study looks at representations of algebraic ideas; one of the essential practices which Blanton et al. (2011) considered for the work on algebraic thinking in the classroom. From the various approaches to algebraic thinking, we focused on the functional approach, which implies "the construction, description, representation and reasoning with and about functions and the elements they are comprised of" (Cañadas & Molina, 2016, p. 212). When working with functions, children can represent the relationship between variables in different ways. Given the characteristics of the task proposed -with paper and pencil-, children basically used pictorial and tabular representations. We were interested in addressing tabular representation as it has not been explored much at these levels.

In this study, we set two research objectives: (a) to describe the written representations used by 5-year-old children to organize the relationships between covarying quantities; and (b) to compare the characteristics of these representations with conventional tables.

REPRESENTATIONS

Among the representations used by children when working with problems involving functions, are: (a) natural-oral language; (b) natural-written language; (c) pictorial; (d) numerical; (e) tabular, and (f) algebraic notation (Carraher et al., 2008). Regarding tables, they can be considered graphic organizations where information is arranged on a double axis ordering and systematizing interrelated data (Campbell-Kelly et al., 2003). Function tables are those which help represent the relationship between two covarying quantities. In a function table there are several distinct characteristics: (a) arrangement of values in rows and columns, following an order such that the independent variable values are on the right and the dependent variable ones on the left (Martí et al., 2010); (b) heading of the columns indicating which variable each one refers to (Brizuela et al., 2021), and (c) separation, with a line or another element, of the independent and dependent variable values (Estrella, 2014).

Several authors have emphasized the importance of studying how children interact with tables as these are tools that support early development of functional thinking (Brizuela et al., 2021; Estrella, 2014). Prior research evidenced that children aged 5-6 used tables to solve problems involving linear functions (Anglada et al., in press; Brizuela et al., 2021).

METHOD

This study is part of a CTE implemented in a class of 24 children, aged five, in Spain. They had not previously worked on tasks involving relationships between variables or tables. This allowed us to observe how they spontaneously approached the building of tables. The CTE comprised four sessions; this paper focuses on the third one. We chose this session because there, after working with a function machine (Figure 1a) involving the function f(n)=n+2, they were given for the first time a blank sheet of paper to freely represent what they had done during the session. Students observed how the machine worked through several examples, and they had to discover the general rule. We used a table (Figure 1b), representing the numbers with material inspired by the cards of Herbinière Lebert.

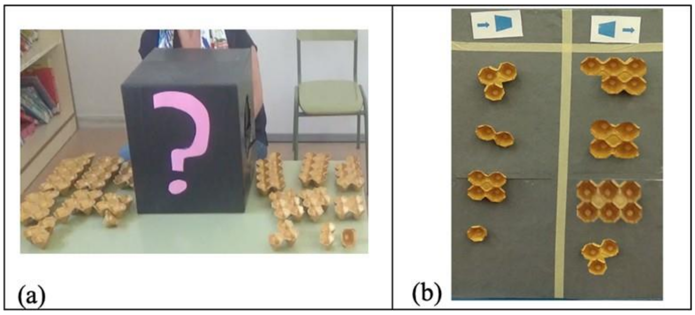

Figure 1: Function machine (a) and manipulative table introduced (b)

At the end of the session, we asked them to explain what we had done. For this, they were given a blank page and a marker. The goal was for them to represent the relationship between the variables.

We analyzed the written productions of the children considering two analysis categories based on our conceptual framework. On the one hand, we considered the category representation of relationships between variables, distinguishing between: (a) evidence explaining the relationships between variables for only one particular case; (b) for various particular cases; or (c) no evidence of relationship; and on the other hand, the category characteristics of tables, identifying whether: (a) they followed the order of the variables; (b) they used headings; and (c) they separated the variables in any way.

RESULTS

Out of the 24 productions analyzed, we identified pictorial representations in 17 of them. Specifically, representations which illustrate diverse aspects not related to the function machine covered in this session, nor did we see evidence of the relationships between the variables involved. In the remaining 7 children, we did identify relationships between variables. These productions are shown in Figure 2.

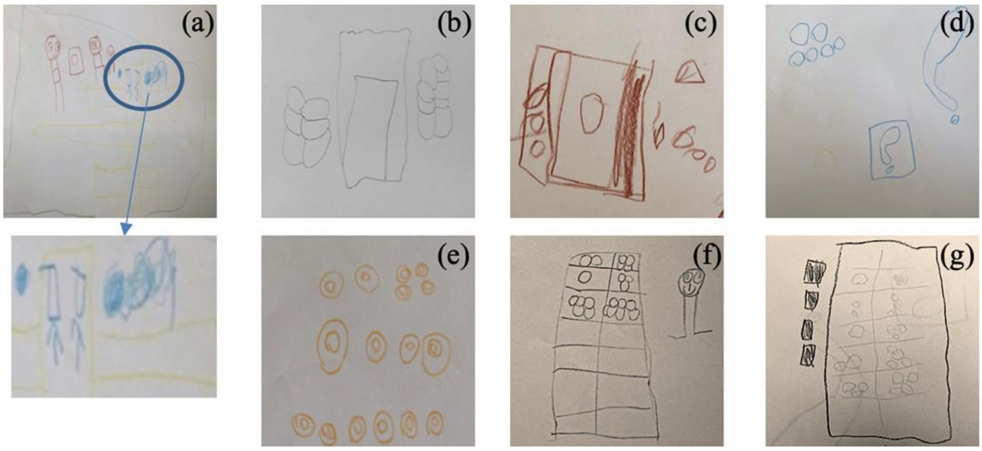

Figure 2: Children's written productions

Regarding the representation of relationships between variables, we identified evidence of a representation of the relationships between variables for a single particular case (see Figure 2, in (a), (b), (c) and (d)) in the answers of four children. For example, in "a", they drew a stone going into the machine, they indicated the machine added 2 and they drew three stones coming out. In addition, the pairs of values represented are correct except for "b", although we noticed the number of stones coming out is higher than the one going in. In three of these representations, "e", "f" and "g", we can see more than one pair of values. For example, in "f" the child represented pairs of values in rows: two stones went in and four came out; one went in and three came out; and five went in and seven came out.

Looking at the characteristics of the conventional tables, we found that in the children's productions, no child used anything equivalent to a heading. Except for "a", they all kept the order of entering from the left and exiting from the right. They all somehow separated the quantities corresponding to each variable, except for "e". In this case, they did leave space between the two cases represented, as well as distributing them in rows. In "f" and "g" they drew the divisions in columns and rows. In both cases, they built function tables that make sense and show their logic, adequately organizing and representing the relationship between the variables. They separated the values both vertically and horizontally and respected the order when placing values in columns.

DISCUSSION AND CONCLUSIONS

Regarding the first objective of the study, 7 of the 24 participating children evidenced representations which related variables, both for a particular case and for several. These results match those in the paper by Anglada et al. (in press), who conducted a similar study with another class of 24 children and a different task.

As for the second objective, the written productions of children highlighted the spontaneous manner in which they represented the relationship between the covarying quantities. The fact that none of them put a heading on the table columns differs from findings in other studies involving children of the same age group (e.g., Brizuela et al., 2021). This could be a result of the manipulative process when interacting with the function machine and the table on an adhesive board where it was clear which order the variables had to be placed in.

In line with Brizuela et al. (2021), the open format to build tables (we gave them a blank sheet of paper), has helped delve into how 5-years-old children relate covarying variables. This study contributes to the knowledge on the process for building function tables by children. We have shown a practical example of how to use function tables with manipulative material before moving on to paper representation.

Acknowledgments

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GENERALISATION IN INTERACTIVE GROUPS. AN EXPERIENCE WITHIN A FUNCTIONAL CONTEXT IN EARLY ALGEBRA

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In this study, we analyze the generalization process carried out by 9- and 10-year-old students working on two linear function tasks in interactive groups (IG). Among the findings, we observe that students show different structures for the same function and generalize in three different ways: algebraically, arithmetically and incipiently. In addition, we observed that the work with IG favors argumentation and explanation by the students, facilitating the attainment of generality.

INTRODUCTION

In the last twenty years, algebraic content has been promoted in the Primary Education stage, not only in the field of Mathematics Education research, but also in the curricula of various countries. This initiative is included in the so-called Early-Algebra current, with the aim of promoting the development of algebraic thinking from the first levels of schooling (Pincheira & Alsina, 2021). Algebraic thinking is concerned with ways of thinking that go beyond algebraic or alpha-numeric expressions (Kieran, 2011). Its basic components are generalization, representation, justification and reasoning (Blanton et al., 2021). This translates into an organization of school algebra from different approaches, which can be grouped as follows: patterns, functions and generalized arithmetic. We focus here on the functional approach. This approach to early algebra addresses inter-variable relationships in functions for which the study of structures and their generalization is essential (Blanton, 2008). When we speak of generalization, we think of the ability to recognize generality beyond particular cases and their representation. Structures are the regularities in the inter-variable relationships present in a function and can be observed when working with both specific cases of functions and their generalization (Torres et al., 2021). From that perspective, generalizing consists in establishing the general structure between covarying quantities.

In the scientific literature, there exist different works focused on the analysis of the basic components of functional thinking, although most of them are based on individual children's activities (Radford, 2003). Therefore, we consider it necessary to go deeper into the analysis of these processes when students interact with each other. For this purpose, the activities designed were developed within interactive groups (IG). IGs are dialogical learning environments in which students of heterogeneous abilities participate, facilitated by an adult who does not necessarily have to be a teacher. The characteristic of IGs is that dialogic learning emerges as a result of egalitarian dialogue. Research tells us that the results obtained with the implementation of IG are beneficial both academically and socially (García-Carrión et al., 2020; Villardón-Gallego et al., 2018).The aim of this paper is to analyze the generalization process that emerges when children are confronted with a task focused on the functional approach within an IG. In the process of generalization, the following questions arise: Do the students see the structure of the function involved? Do they manage to generalize it? If so, how do they express the generalization? Do IGs foster functional thinking?

METHOD

This qualitative, exploratory and descriptive study consisted of a classroom teaching experiment within the research-design paradigm (Bakker, 2019). Four sessions were conducted with IGs, consisting of 3-4 children aged 9-10 years, and lasting 20 minutes. For each IG, a questionnaire was applied with questions starting from particular cases, followed by distant and/or indeterminate cases and leading to generalization. The context used in the questionnaire was a ball machine in which balls go in and out following two different functions: f(x)= x+2 and f(x)= 2x. The task statement was: "A mystery box has arrived at the school. It changes the number of balls you put inside. Your task is to find out what change the machine makes. Look at what the following machine does and answer the questions". Before moving on to the questionnaire, some particular cases were presented for the students to identify the structure involved between the variables of the task, see figure 1.

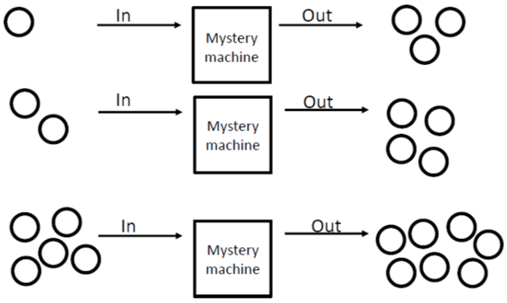

Figure 1. Example of the particular cases given for the function f(x)= x+2

The questions in this phase were of the following type: how many balls would come out if the machine input 10 balls? and if 100 balls come in? Explain your reasoning. After working with the particular cases, the adult volunteer asked the following question for the general case: What is the machine doing? In figure 2, an example of a task included in the questionnaire is presented where different particular cases and others referring to indeterminate quantities (some balls) appear.

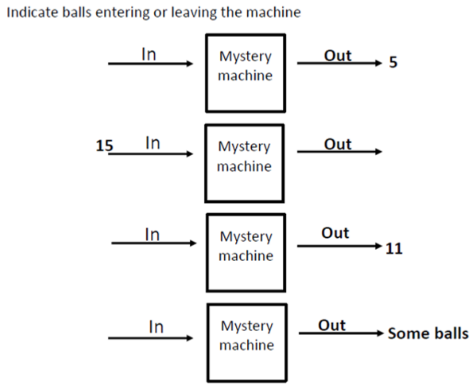

Figure 2. Example of a task from the questionnaire applied for f(x)= x+2

Children's verbal responses in the IG were analyzed in terms of two components of functional thinking during the generalization process: a) structures evidenced and b) types of generalization.

RESULTS AND DISCUSSION

Two distinct phases can be distinguished within the IGs and in the generalization process carried out:

a) Phase 1. Students give their individual views on how the machine works. This is a preliminary phase in which students observe the first examples of particular close cases and express their first ideas. Here the students stay in a trial-and-error phase, experiment and count on their fingers to express a conclusion about what the machine does.

b) Phase 2. This phase is where the interactions in the group take place and the work with different distant and/or indeterminate cases takes place. Once they have made their first conjectures, they check them on the basis of what the other members of the group say. We see this in the following excerpt, an example of working with the function f(x)= x+2:

Volunteer (V): ok, let's suppose that "¿" balls come in, how many would come out?
E1: can I put 8?
V: and if I tell you that it doesn't have to be 8, it can be any number, how do you do it?
E8: or it can be 5 or 10.
V: could you pass this relation (pointing to the paper, referring to "?+2").
E8: I think so, what happens is that it could be 4 and come out 4+2.
E10: No, I don't think so. Because if you don't know what he has put in the machine and this ("?+2") has come out, it would always be the same, so no.
E1: Yes, but "¿" is any value, I think it could be.

With these differentiated phases, the structures evidenced by the students throughout the process towards generalization and the expressions they use to generalize within the IGs were analyzed.

Structures evidenced during the generalisation process and expression of generalisation

Different structures were evidenced for the function f(x)=x+2 when working with the particular cases. The structures identified were of three different forms (y=x+2, y=2x (incorrect for the context) and y=x+1+1), see table 1. For the function f(x)= 2x the structures evidenced were y=2x and y=x+x. Some of these structures were generalized throughout the process. Note that symbolic notation was used to represent the structures evidenced by the students although they expressed them in verbal representation.

Function	Structure	Verbal representation of generalizations	Type of generalization
F(x)=x+2	y=x+2	The machine always produces two more balls	Algebraic
	y=2x	The machine is multiplying, 2 plus 2 is 4.	From individual cases
	y=x+1+1	The machine is adding. If I add 1 to 8, it would be 9 and if I add 1 to 9, it would be 10.	From particular cases
F(x)= 2x	y =2x	The machine produces more balls. It will multiply.	Incipient
	y=x+x	You add 1 to 1 and you get 2, you add 2 to 2 and you get 4 and so on... I've added it up. Adding the same number	Algebraic

Table 1. Structures and generalizations identified.

With respect to the structure y=2x, which was shown to be erroneous in the work with the function y= x+2, it happens that the two functions used in the study give the same output if the input to the machine were two balls. This is a fact to be taken into account in the data collection.

As far as generalization is concerned, it occurs in different ways depending on the degree of sophistication of the answers. We find: a) algebraic generalizations supported by temporal deictic (always and so) which indicate the abstraction of the regularity and an awareness of the structure involved; b) incipient generalizations when the pupils describe the structure by mentioning only the arithmetical operation or express the relation without mentioning the variables and c) arithmetical generalizations sustained from examples of particular cases worked on.

CONCLUSION

It is concluded that in phase 1, the students do not take into account the interventions of the other members of the group; a trial-and-error phase takes place where individual reasoning takes place in which the first conjectures about the implicit structure are formulated. In other words, the first relationships become evident. In phase 2, some of the groups agree on their answers by means of explanations and arguments. In the interaction with the arguments, the students can corroborate or refute the previous conjectures (phase 1) they had obtained and can reach the generalization of the structure. The generalizations observed in the IGs have been algebraic, arithmetic and incipient. The methodological design and group work is encouraging ways of representing indeterminacy through the application of symbolic indeterminate terms (example: "¿+2", in symbolic form) and ways of arguing or explaining new relationships or regularities between two covariant quantities.

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ALGEBRAIC THINKING AND LEARNER`S SECOND LANGUAGE STUDENTS: IMPORTANCE OF THE ANALYSIS OF GESTURES

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Numerous studies have explored the significance of gestures in students' algebraic thinking. In this article, we propose that examining the gestural repertoire becomes particularly pertinent when scrutinizing the algebraic thinking of second language learners. To illustrate this perspective, we present the case of Daniel, a third-grade Haitian student enrolled in a Chilean school where a teaching experiment took place. Following the experiment, we administered a written assessment and conducted an interview to assess Daniel's algebraic thinking. By analyzing a semiotic repertoire inclusive of gestures, we discovered that Daniel demonstrated algebraic thinking, contrary to our initial inference drawn from the written test. These findings underscore the importance of considering the interplay of diverse semiotic resources when evaluating the mathematical development of second language learners.

INTRODUCTION

Gestures play a relevant role in communication and learning, as they often convey information not present in the speaker's discourse and can reveal implicit knowledge (Novack & Goldin-Meadow, 2015). Likewise, from a multimodal perspective of cognition, gestures are an integral part of the set of semiotic resources that students mobilize when learning and communicating mathematical concepts (e.g., Arzarello, 2006; Radford, 2009; Ng, 2016). In this sense, gestures can be understood not only as a communicative element that allows ideas to be expressed, but also as a way of organizing thought in the problem-solving process (Kita, 2000). Specifically, in the development of algebraic thinking, gestures, in coordination with other semiotic means, are a key aspect in the early stages of forming concepts such as variability, before such concepts can be communicated in culturally more conventional formats, such as symbolic notation (Radford, 2009; 2018).

On the other hand, literature specializing in second language acquisition has emphasized the importance of non-linguistic forms of communication. These students, who often possess a limited vocabulary, rely more heavily on gestures as a means of supporting discourse (Ng, 2016). Simultaneously, second language speakers employ gestures as a self-organized form of mediation; that is, gestures coordinate thought and expression in a second language (McCafferty, 2004). The challenge of learning mathematics while attempting to master a second language may contribute to explaining why second-language learners often achieve poor results in mathematics, particularly in the case of learning algebra (Morton & Riegle-Crumb, 2019).

Despite the connection between these topics, the manner in which the use of gestures in algebraic thinking could contribute to understanding mathematical ideas in bilingual students has been sparsely investigated. In this research, we address this topic by analyzing the algebraic thinking of Daniel, a Haitian student enrolled in the third grade in Chile. Specifically, we examined his performance at the end of a 6-session intervention, during which students in his class learned to generalize patterns. We administered a written test to assess the students' learning outcomes. To better understand their ideas, we conducted a filmed interview where they explained the meaning of their written responses. Daniel's case was particularly noteworthy, as unlike his Spanish-speaking peers, we observed significant differences between his performance in the interview and on the test. Considering this context, this study aimed to delve into these inconsistencies. To do so, we analyzed the test and the interview video to determine how written, spoken, and gestural registers contribute to identifying the algebraic thinking of a second-language learner student.

Algebraic thinking

According to Radford (2018) algebraic thinking is defined by three fundamental aspects: a) it relies on indeterminate quantities, b) these indeterminate quantities are manipulated in an analytical manner, meaning they are operated on as if they were known, and c) they are represented using various semiotic resources. These resources include alphanumeric language, but also less sophisticated ones like gestures. Radford (2018) termed the stage where students identify key aspects of the sequence and apply them to specific numeric terms as *factual*. During this stage students use semiotic resources such as rhythm and gestures to communicate variable and non-variable aspects of the sequence, which they perceive but are not fully able to verbalize. In the second stage, referred to as *contextual*, students can verbalize the generalization, explicitly referring to indeterminates quantities. It means, students can answer questions that require them to explain how they found the number of elements corresponding to *any* figure in the sequence. Finally, at the *symbolic* level, these resources are replaced by more sophisticated and precise semiotic systems like alphanumeric language.

Therefore, an important aspect when analyzing students' algebraic thinking is to identify whether they are capable of working analytically with indeterminate quantities, taking into account the semiotic means they mobilize. In this study, we focus on the factual and contextual stage, considering the way of mobilizing bodies, including gestures and calculations, such as written or spoken natural language.

METHODOLOGICAL DESIGN

A single case study was conducted. In this study, we focused on Daniel, a third-grade student from Haiti enrolled within a Chilean school. Like many Haitian children in Chile, he attends a school serving a low-income population and received a low score on the State mathematics test (Gelber et al., 2021). Haitian Creole is spoken in his household, and both he and his family have limited proficiency in Spanish. Daniel's class participated in a six-week intervention. The goal of the intervention was to offer the opportunity to address various contextualized problems whose variables were related to linear functions. The students generalized the observed relationships, applying them to different terms in the sequence, and described these relationships in natural language and through variable notation. At the end of the intervention, a written test was administered based on the problem shown in Figure 1.

Maria is arranging the tables and chairs where the guests at her party will sit. She has several small square tables that will be arranged in a row so that everyone can sit together. If she puts only one table that seats 4 people. If she sets 2 tables, 6 guests fit and if she sets 3 tables, 8 guests fit.

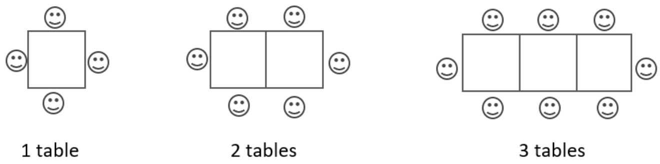

Figure 1: Problem used in the test

The test comprised eight questions, including calculating the dependent variable for small and large quantities, organizing data in tables, and explaining the generalized rule using both natural and symbolic language. In this study, we focus solely on two of the questions:

1. How many guests can sit if 204 tables are set?
2. Describe how many guests can sit for any number of tables.

Students answered all the items on the test individually and without assistance. Additionally, the day after the test, students were interviewed and filmed to delve deeper into the reasoning they employed when responding. The objective of these interviews was to establish whether the conclusions derived from the written responses aligned with the reasoning the students claimed to have had. To achieve this, we provided each student with their written test and asked them: How did you arrive at your answer?

Two different analyses were conducted. The first analysis exclusively involved the test, relying solely on written records. The second analysis was performed by considering the interview video, employing a semiotic analysis of multimodal activity (Radford & Sabena, 2015). In this context, we took into account spoken and written language, diagrams or drawings, calculations, gestures, and any actions undertaken by the students when explaining their responses. We applied the same categories for both analyses. For question A, we observed whether the response suggested that the students had identified the correct structure of the task and can be applied to term 204. In other words, we assessed if the answer revealed the structure $y=2x+2$ (or an equivalent, such as $y=x+x+2$), regardless of whether they arrived at the correct answer, or if they identified part of the structure (e.g., $y=2x$). For question B, we examined whether the students described a generalized rule, it means, without referring to specific terms that would allow obtaining the quantity of guests for any number of tables.

RESULTS

Written responses analysis

Upon reviewing Daniel's test, we found that he did not answer the first question correctly (Figure 2A). In his response, it can be inferred that he dismissed several calculations, for which he likely used operations like 204x2 and 204+2, ultimately arriving at the incorrect answer of 500. However, it is not clear which operations he performed to arrive at this result. From the above, we infer that Daniel did not apply the operational structure to a remote case. To address the second question (Figure 2B), he included only one numerical example, demonstrating some understanding of the operations. However, he did not describe the relationship between variables in general terms.

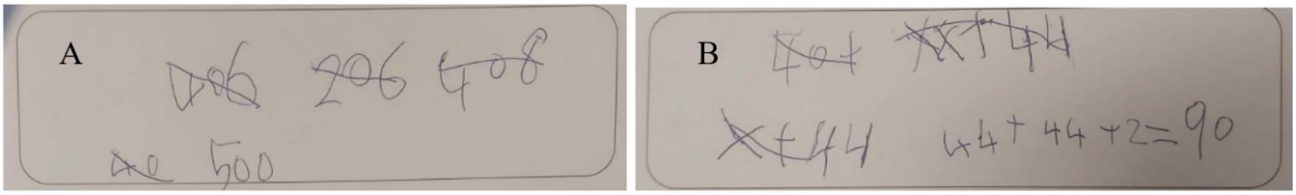

Figure 2: Daniel's answers to questions 1(A) and 2(B)

Semiotic multimodal analysis

During the interview, we asked him how he had arrived at his answer. In the following dialogue, he explained how he answered the first question:

Daniel: I made a mistake, it was very difficult. I first wrote 406, and then 206, and then 408, but I was about to put another 400…and I can't remember what number, but I then took it away and saw that it was 500

Researcher: Ok, but what did you do to obtain 500?

Daniel: (Thinking silently for 5 seconds) Ah! because like that under the table, and when I was left with 800, I added 2 and was left with 500.

When Daniel said the last sentence (which doesn't make sense in Spanish), he moved his hand horizontally twice from left to right, indicating the row of tables (Figure 3A to 3D). When he said "two," he placed his hands on each side (Figure 3E).

Figure 3: Daniel's gestures when he responds

In this fragment, it can be seen that Daniel physically enacts the description of the key elements of the situation: he passes through the row of tables twice and indicates "two" with both his words and hands. Indeed, for any term in the sequence, the problem is solved by considering the number of tables twice and adding two. Daniel encounters difficulties both in performing the involved calculations and in verbalizing the procedures he performed. In fact, the first part of the structure (adding the rows twice) is expressed in his gestures rather than in his speech. That is to say, there is no observed coordination between speech and gestures in this part. Therefore, in this part of the interview, paying attention to his gestures is crucial to interpret that Daniel can identify the key elements of the situation.

Subsequently, we asked to explain in more detail what he had added. He responded, "200 + 200, 400, and 4 + 4 + 2, is…well…it makes…more, and you added to the 400 to make 500". When he said this sentence, Daniel leaned forward and hid his hands under the table. Here, Daniel has managed to explain his calculations better, using the Spanish language. Daniel added 200 + 200 (i.e., the hundreds of 204+204), and then added only the units 4 plus 4, plus 2 from the sides. It is probably when he added 2 to 8, instead of increasing the tens, he increased the hundred and reached 500. This, added to his previous gestures, confirmed our assumption that he had indeed identified the structure despite his errors in calculation. The calculations he performs for term 204, although arithmetically incorrect, show that he perceived the sequence and could apply it to any remote specific term. It is curious that when articulating this sentence, he hid his hands. Possibly, the gestures previously made allowed him to organize his thoughts, so now he could articulate a spoken sentence without gesturing.

Subsequently, we asked him how he had arrived at his answer for the second question. He pointed out that he thought it had to be solved by putting down symbols, so we posed the question to him again. Daniel responded:

Daniel:	Add to what is above and what is below and what is on the side.
Researcher:	And how much is above, below, and on the side?
Daniel:	The one above and the one below are the same, the one next to them are only ones.
Researcher:	And those numbers that are the same, what do they correspond to? To the number of guests or tables?
Daniel:	Of guests... of tables... because if there are 44 tables, there are 44 guests above and below too, but on the sides there are not 44, there is only 1.

In this video fragment, he no longer moves his hands and expresses more clearly what he has managed to identify in the problem: the constant "on the sides there are not 44, there is only 1," and what changes "the one above and the one below are the same." It is evident that he understands the structure of the sequence and can apply it to any term. Even though he has not yet been able to express it in symbolic language, it can be inferred that Daniel shows factual algebraic thinking, as an understanding of variability can be perceived in the actions he performs. He is beginning to acquire the contextual level, as he has started to describe the relationships in a generalized manner, although he is still resorting to the use of specific numbers to exemplify them.

Our conclusions drawn from the written exam and the broader semiotic repertoire, which included gestures and spoken language, were notably different. Only when considering gestures and spoken language could we truly understand what Daniel had learned throughout the intervention.

DISCUSSION

While the importance of gestures in the analysis of algebraic thinking has been widely discussed (Radford, 2009; 2018), how this manifests in students who do not possess language proficiency equal to their peers has been scarcely examined. The case presented in this research serves to illustrate that including gestures in the analysis of algebraic thinking is especially relevant when students have limited command of a central semiotic resource, such as language. Although Daniel could communicate and understand most ideas in Spanish, he was naturally less fluent than native speakers. While answering the first question, he made clear gestures that denoted an understanding of the structure of the sequence, although his verbal repertoire was incomprehensible. Therefore, the coordination between both registers in this part was less evident, unlike what has been observed in the case of native speakers, where the different semiotic means are usually coordinated (Radford, 2018). In a second attempt to explain his procedures, Daniel was able to better articulate a spoken explanation of his calculations. It is possible that the gestures previously made allowed him to organize his thoughts to perform the demanding task of expressing his ideas more clearly in Spanish. A finding that seems to be in tune with what is suggested by McCafferty (2004). Finally, in the second question, he seemed to better explain an answer for the indeterminate case, restricting the use of gestural communication. Surely Daniel's level of mastery of Spanish, although limited, allowed him to move from gestural to spoken register. This constriction of semiotic resources may be similar to what is described by Radford (2018) for first-language speakers.

In addition to facing linguistic difficulties, Daniel also had significant arithmetic challenges. These difficulties resulted in his poor performance on the test. This aligns with the observed low performance in mathematics and other disciplines of in the population of Haitian students in Chile (Gelber et al., 2021). If we had solely relied on written responses, Daniel's mastery of skills that can be considered tangential to algebraic thinking, like calculation and proficiency in Spanish, might initially have suggested a lack of competence in the test tasks. From an evaluative standpoint, differentiating the attribute being assessed, in this case, the ability to analytically approach a sequence, from other non-central skills is an important aspect when assessing for pedagogical or academic purposes (AERA, APA & NCME, 2014). This provides students with the opportunity to demonstrate what they have learned more fairly and accurately. Considering a broad semiotic repertoire could benefit second-language learners, providing them with the chance to showcase their algebraic knowledge.

A limitation of this study was that, in order to delve into details, we focused solely on Daniel's performance in two of the questions comprising the final test. An analysis of how he approached the remaining questions on the test could help to understand his overall performance more comprehensively. Similarly, comprehending the way he mobilized various semiotic resources throughout the entire intervention could further characterize his learning process. Delving deeper into this topic, from a semiotic perspective, could contribute to understanding the development of learning in second-language learners.

Acknowledgments

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CONNECTING EARLY ALGEBRA AND SUSTAINABILITY THROUGH DILEMMAS IN ALGEBRAIC WICKED PROBLEMS

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Early algebraic thinking can be connected in significant ways to sustainability issues through algebraic wicked problems (AWP). AWP are problems that mobilize children to think algebraically while experiencing dilemmas on values at stake in sustainability. Both algebraic thinking and sustainability issues become inseparably connected. From an ongoing analysis of 10 research group meetings over a school year in a group consisting of teachers, pre-service teachers, and researchers, and based on observed lessons, we explore such dilemmas as a central characteristic of AWP. This provides details into what may be important aspects in the further design of AWPs.

INTRODUCTION

As society gets more complex and even young children are confronted with the challenges of sustainability and climate change, it becomes necessary to make explicit connections between mathematics teaching and learning, such as the development of early algebraic thinking, and the ways in which such early algebra plays a role in preparing children to address current societal and environmental difficult situations. Barwell et al. (2022) highlight how the current sustainability crises — in environmental, social, and economic terms — do not get enough attention in mathematics education. Solares-Rojas et al. (2022, p. 204) argue that young generations need to "build tools, options, and paths to prepare /…/ to confront" these issues, and that educational strategies need to be developed to prepare pupils to face these issues. In this line of thinking, the concern of mathematics education for "the socio-ecological" (e.g., Coles, 2023) cannot be limited to the use of data concerning sustainability, climate change or real-world 'environmental impact' as a context in the teaching of mathematics (e.g., Boylan & Coles, 2018). Pupils also need to be challenged to go "behind the data", and in reflective dialogue discuss possible underlying structures/values of the data (Barwell, 2013). The current predicaments of sustainability invite us to push the boundaries of the significance of mathematics education for new forms of critical citizenship.

In our current research work, we particularize these challenges to early algebra. Our previous review of early-algebra research pointed to the dominance of studies that justify the need to develop early algebraic thinking in terms of their contribution to the logical and cognitive development of pupils (Fred et al., 2022). It also showed a lack of studies about early algebra in connection to cultural and societal issues such as sustainability. In an attempt to tackle such shortcoming, we have explored the possibilities of working toward early algebra teaching and learning that takes the challenges of sustainability as previously described (Fred et al., 2023a).

For this purpose, Jenny Fred built a research team consisting of four grade 1-3 teachers, four pre-service teachers, and three researchers (the three authors of this paper). We collaborated (Fred et al., 2023b) on the process of inventing, designing, planning, and reflecting on a new type of problem for early algebraic activity: Algebraic Wicked Problems (AWPs). The collaboration around the AWP took place during one school year, during which 10 research group meetings were closely documented and studied.

In our explorative work of creating both the notion of AWP and formulating concrete examples, we drew on Rittel and Webber's (1973) idea of a wicked problem as a rhetorical, creative, and critical/emancipatory tool (e.g., Lönngren & van Poeck, 2021). Already in the early 1970s, the notion of wicked problem was used to refer to complex, multidimensional predicaments that are not easily solvable by simple policy intervention; indeed, wicked problems may not even be "solvable" at all (Rittel & Webber, 1973). The term was introduced to problematize what they called 'tame problems' in the natural sciences, in contrast to complex societal-technological-scientific problems, one of which can be the multi-layered, multilevel problem of sustainability and climate change in current times. Building on this idea, science educators for some time (e.g., Lundegård & Caiman, 2019) — and mathematics educators more recently (e.g., Steffensen et al., 2021) — have used the notion of wicked problems to create situations of teaching and learning that go beyond many of the 'tame problems' used in typical school word-problems. The usual problems that pupils meet when working with algebra, such as solving an equation without context or figuring out the general rule for a number pattern, can be characterized as tame problems. In a tame problem it is clear what the pupils should do, and it is also clear "whether or not the problems have been solved" (Rittel & Webber, 1973 p. 160). In contrast, wicked problems can never be definitely solved or, at most, are re-solved repeatedly because they involve multiple dimensions in connection, values mobilized, and viewpoints at stake.

The explorative work in the research team consisted of taking on board the notion of wicked problem to reinvent algebraic problems for early school algebra. Furthermore, if the challenges of sustainability were not to be taken as a simple context of formulating tame school algebra problems, but as a possibility to connect algebraic thinking as a significant tool to explore the wickedness of sustainability, then pupils should be invited to integrate algebraic thinking and sustainability in their work with the AWPs. Algebraic thinking should work as a resource or a tool for concrete critical thinking and action (Skovsmose & Valero, 2008), as an analytical tool for exploring models of "reality", and for creating an awareness of how those models only visualize certain aspects of a reality and leave other unnoticed (Cai & Knuth, 2011; Kieran, 2018). That is, the pupils should be invited to explore, interpret, and reason about functional relationships between quantities in given models of pupils' "reality" (e.g., Coles & Ahn, 2022).

How to make algebraic thinking and sustainability issues becoming simultaneously integrated in pupils' reasoning is not an easy matter. In our try-outs of different formulations of AWPs with pupils, although the framing of an AWP consisted of these two aspects, the pupils did not necessarily connect them with one another. In our further work on that challenge, we have experienced how dilemmas can open a possibility to make the merging of algebra and sustainability happen. Dilemmas are built-in incompatible condition in a problem which requires pupils to analyze and reflect on quantitative data in a model of reality (Davydov, 2008). This paper explores dilemmas and their potentiality for connecting early algebraic thinking and sustainability simultaneously in pupils' work with AWP, and for further designing AWPs.

METHODOLOGY

We drew on multiple sources of data that all are related to the research team's invention, design, planning, and reflection on AWPs.10 research team meetings that consisted of planning and reflecting on working with AWPs were recorded and transcribed; field notes of lesson observations (here we particularly report on grade 3 lesson for about 20 pupils). Transcribed research team meetings were marked to indicate whether the particular focus of the conversations were involving: aspects of algebra and algebraic thinking, aspects of teaching and learning, aspects of sustainability, or aspects of the purpose of education for critical citizenship. The data as a whole was then analyzed to understand the instances where the aspects of algebraic thinking and sustainability overlapped, and how they became integrated in the pupils' work with AWPs as well as in the discussion in the research team. For this paper we have chosen to concentrate on two concrete examples to tease out how dilemmas emerged and turned out to be an important element in thinking what may characterize AWPs.

In what follows, we introduce a situation of work with an AWP that was constructed in the research team. Then, we provide two descriptions of lesson fragments where pupils work on AWPs. One of the descriptions exemplifies how a dilemma emerges but does not connect algebraic thinking and sustainability; while the second description shows the opposite. The descriptions are structured to highlight: 1) the built-in incompatible condition emerging, 2) the setting in which it emerged, and 3) how algebraic thinking and sustainability related in the situation. Finally, we discuss some central dimensions of dilemmas that we have identified so far. Our data suggests that these dimensions may be helpful in further designing AWP where algebraic thinking and sustainability intertwine.

EMERGING DILEMMAS

The teacher starts the introduction of the situation by asking the pupils if they know what carbon footprint is. One pupil answers that it relates to environmental damage as for example waste, carbon dioxide and exhaust fumes. The pupil continues this explanation by stating that this kind of environmental damage is what causes the Antarctica to melt which, in the future, will make the worlds water level to rise. In this introduction, the pupils are also invited to a short discussion concerning what a budget is and how you cannot "buy" more than your budget. Finally, the pupils are asked to create a joint carbon footprint budget, in pairs, with things/toys they want for Christmas. The pupils are told that they can only choose things/toys from a given carbon footprint table.

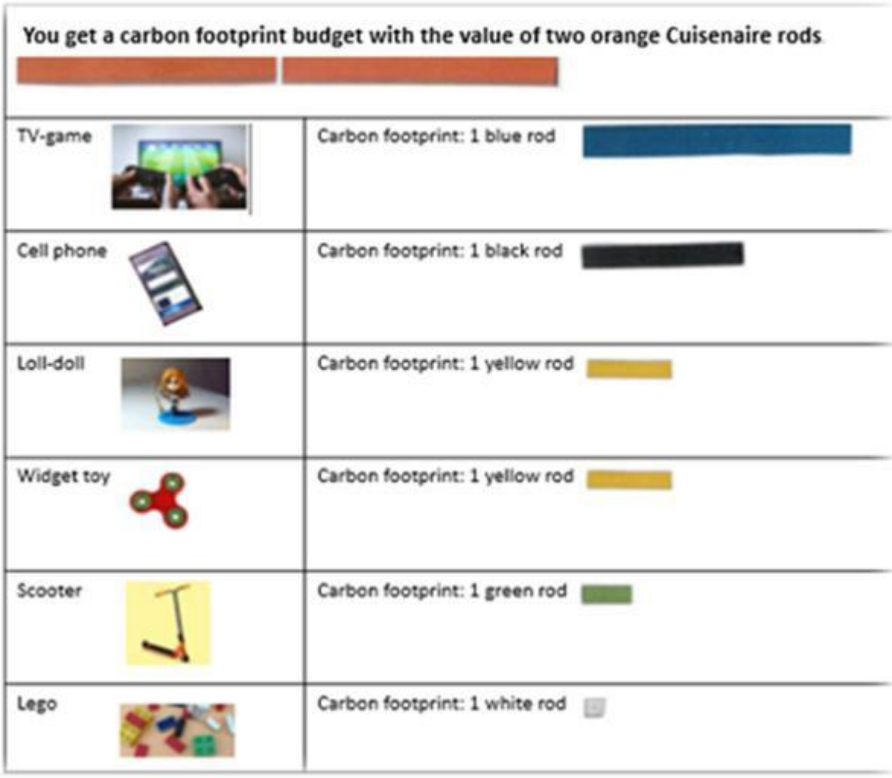
Example 1: How much Lego?

The *built-in incompatible condition* in this example consists of a value conflict between Lego's economic value and its carbon footprint value: Lego is expensive to buy but has a low carbon footprint value.

In setting the scene the teacher introduces the idea of a carbon footprint budget that pupils have at their disposal. It consists of two orange Cuisenaire rods (C-rods) that the teacher shows to the class (Figure 1). The teacher continues presenting the pupils with a carbon footprint table of different things/toys (Figure 2), and explains that the things/toys in the table have been given different values in terms of their carbon footprint, represented by different C-rods. When explaining the table, the teacher shows the C-rod that represents the Lego's carbon footprint value by showing one white C-rod to the class (Figure 3).

Figure 1: The teacher showing the budget (reconstructed picture)

Figure 3: The teacher showing the Lego's carbon footprint (reconstructed picture)



You get a carbon footprint budget with the value of two orange Cuisenaire rods

TV-game	Carbon footprint: 1 blue rod
Cell phone	Carbon footprint: 1 black rod
Loll-doll	Carbon footprint: 1 yellow rod
Widget toy	Carbon footprint: 1 yellow rod
Scooter	Carbon footprint: 1 green rod
Lego	Carbon footprint: 1 white rod

Figure 2: The carbon footprint table

A *value conflict* emerges around the question if the C-rod represents the value of one Lego brick or a whole set of Lego bricks.

Pupil 1: What is one [emphasizes is and one] Lego?
Teacher: This is probably a box of Lego. Not just a piece but... What do you say? What do you think?
Pupil 2: One Lego bag?
Teacher: One Lego bag? And...
Pupil 1: I think it is one Lego set.
Teacher: Yes, one Lego set. But not the biggest.

The exploration, interpreting and reasoning above concerns primarily the relationship of correspondence between the given value of one C-rod and the amount the Lego (one brick or a package of bricks). The discussion does not connect to any sustainability aspect. Instead, it seems that the potential sustainability issues that could be connected to the amount of Lego remain working as part of an unreflected context. The focus here is on quantities and their correspondence in value.

Example 2: How come the cost of lol-dolls, Lego and scooters?

The *built-in incompatible condition* in this second example is that, in the table (Figure 2) the value of a toy/thing is expressed in the currency of carbon footprint while pupils are familiar with currencies for economic value. Also, the value in carbon footprint terms does not seem to align with the toy/thing' economic value. That is, an "expensive" toy/thing in money is "cheap" in carbon footprint.

In setting the scene, the teacher asks the question: "Does the table feel fair?" and simultaneously points at the carbon footprint table (Figure 2), that is displayed on the whiteboard in front of the pupils. A value conflict emerges when the pupils start, as a response to the teacher's question, to explore, interpret, and reason about which characteristics of the objects are affecting the toys/things given values in the carbon footprint table.

Pupil 4: That little lol-doll can't be much more than Lego, can it?

Pupil 5: And then the scooter... Scooters are big...

Pupil 6: But you have a scooter for a really long time... And there are a lot of other things than just plastic.

Pupil 6's expression "you can have a scooter for really long time", can be recognized as an awareness of durability of a toy that is important for sustainability. As part of the series of comments, it also provides an explanation for the corresponding relationship between the scooter's and the lol doll's carbon footprint value. We interpret such expression as a manifestation of simultaneous integration of algebraic thinking and sustainability issues emerging in the pupils' conversation. Here, in the line of reasoning among pupils, pupil 6's expression seems to point to a connection between the materials of the toy/things, their durability and their value in the currency of carbon footprint, to add to the relationships that other pupils are putting forward.

EXPLORING CENTRAL DIMENSIONS OF DILEMMAS

Our conceptualization of dilemmas in an AWP takes its departure in Davydov's (2008) notion of contradiction. That is, the dilemma is seen as a built-in incompatible condition consisting of some kind of value conflict between two incompatible values. It is the value conflict that can invite pupils to explore, analyze and reflect on corresponding relationships between quantitative data in a model of reality. However, to invite the pupils to act on the value conflict, it is also important that it is pupil-oriented and sometimes it is necessary for the teacher to guide the pupils' awareness to the value conflict by actions such as questions or statements.

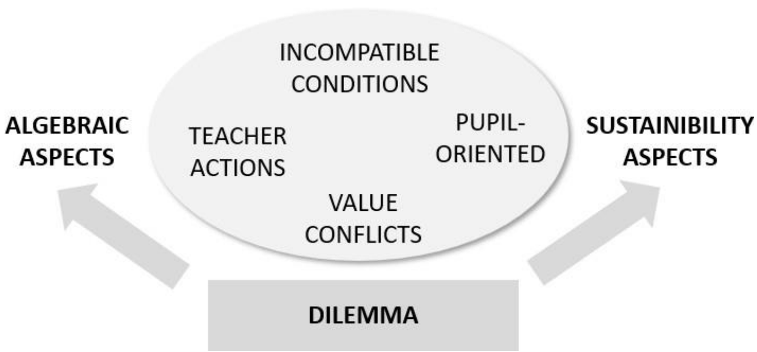


Figure 4: Central dimensions of dilemmas

In Example 1, the built-in incompatible condition consisted of a value conflict between the quantity of Lego bricks and their economic value and its carbon footprint value. The pupil-orientation was two-dimensional. The first dimension was how, in the experience of these Swedish pupils in age 7-9 often want Lego as a gift for Christmas or for their birthdays. In the context of these children, Lego is a toy/thing that they are familiar with and that they desire. The second dimension was the economic that Lego is expensive to buy, but it has a lower carbon footprint value with respect to other toys/things in the table. However, when the pupils were exploring, analyzing and reflecting on potential relationships of corresponding values they were only focusing on the Lego's economic value and its size in terms of amounts of bricks — one brick or a set of bricks. In other words, the pupils' reflections did not involve a consideration on sustainability. The conversation was left unchallenged. In this case, there could have been the opportunity for the teacher to ask questions or make statements that directed the pupil's awareness also towards the Lego's carbon footprint value and thereby to a consideration of sustainability.

In Example 2, the built-in incompatible condition consisted of a value conflict between the lol-doll's, the Lego's and the scooter's value in terms of carbon footprint and their economic value. This example also had a two-dimensional pupil-orientation. First, pupils aged 7-9 in Sweden often play with this kind of things/toys and they are also aware of their economic value. Second, the conflict between the economic and the carbon footprint value is present in, for example, the cheap economic value of lol-doll in contrast to its high carbon footprint value, or a scooter that is expensive to buy but has a low carbon footprint value. The value conflict in this example together with the teacher question "Does the table feel fair?" invited the pupils to explore, analyze and reflect on corresponding relationships between the things'/toys' values in the table. The corresponding relationships that the pupils were establishing and reflecting on involved a consideration for sustainability concerning the durability of different materials to assign the things'/toys' values.

In our exploration of the situations that unfolded in the classroom, we recognized that there emerged dilemmas as pupils thought algebraically to establish connections between quantities, objects, and their economic and carbon footprint values. The dilemmas were related to the built-in incompatible condition as causing value conflicts, which made possible for algebraic thinking to unfold and connect significantly to sustainability.

Our analysis of the situation allows to distinguish that the dilemma and the value conflict are connected through the incompatible conditions that are built in the carbon footprint table (Figure 2). The incompatible conditions concern how the values of the different things/toys in the table are incompatible within one another in terms of economic value. By using a table as a model of reality within a sustainability pupil-oriented context, containing value conflicts, algebraic thinking can become mobilized as an analytical resource or tool to explore, interpret, and reason about corresponding relationships between quantities in the given model (e.g., Coles & Ahn, 2022, see also Fred et al., 2023a, 2023b). Also, the use of C-rods to represent the values seemed to be work as schematics or psychological tools which allowed the pupils to argue and reason about indeterminate quantities analytically, by using a visible and concrete artefact. This has been already documented as an important aspect of promoting algebraic thinking (e.g., Dougherty, 2008).

CONCLUSION

So far in our work, we have started the work of imagining and formulating other types of algebraic problems that invite children to connect in significant ways early algebra (grade 1-3) and sustainability. We have proposed the notion of AWP as we have invented, designed, planned, and reflected on the characteristics of situations that develop pupils' algebraic thinking to "empower children's critical reflection on current, socially relevant issues such as climate change [sustainability], and contribute to a democratic participation in communities of peers and in society" (Fred et al., 2023b, p. 2; see also Hauge & Barwell, 2017). We have found that this is not an easy task! In this paper we have suggested that dilemmas can be a way of working with this challenge. We have also visualized the multidimensional character of dilemmas by pointing at some central dimensions that we have identified so far. Thinking about how situations may be designed around built-in incompatible conditions is an initial guide to create new AWPs. In our further work with the notion of AWP we are interested in inducting new situations and doing more try-outs with pupils to deepen the understanding of dilemmas.

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"IT MADE ME FEEL LIKE AN EXPERT": DEVELOPING FUNCTIONAL THINKING THROUGH CULTURALLY LOCATED TASKS

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Success in algebra plays a major role in equity and lifelong opportunities well beyond the mathematics classroom. High failure rates in algebra see many non-dominant students excluded from equitable higher education, career, and economic opportunities. This paper reports on a research project that explored how non-dominant students in New Zealand developed algebraic thinking through culturally located tasks. Student participants in the study were aged between 10 and 12 years and were of indigenous Māori and Pacific Nations ethnicities. Design-based research and qualitative methods of data collection and analysis were used in the study. Findings revealed that when Māori and Pacific students were given opportunities to draw on their cultures to make sense of functional relationships, they constructed increasingly sophisticated and abstract representations and generalisations of growing patterns, and both their mathematical and cultural identities were strengthened. The findings highlight that providing opportunities for non-dominant students to learn algebra in ways that they see as relevant to their cultures and communities is a powerful catalyst for promoting equity.

Algebra is considered critical to students' mathematical development and has been labelled as a gateway to academic and professional success (Knuth et al., 2016; Morton & Riegle-Crumb, 2019). However, for many non-dominant students, algebra is not a gateway to opportunity; it is a gatekeeper. Inequitable opportunities to succeed in learning algebra exclude many non-dominant students from economic, citizenship, higher education, and career opportunities - particularly those related to the twenty-first century STEM subjects of science, technology, and engineering (Knuth et al., 2016; Morton & Riegle-Crumb, 2019).

Evidence from research studies highlights that young children are capable of sophisticated sense-making of functional relationships and representing and generalising these relationships in diverse ways (Blanton et al., 2015). However, in contrast to the research, Māori and Pacific students are over-represented among lower attainers and under-represented among higher attainers in national and international measures of mathematics achievement (e.g., Educational Research Unit and New Zealand Council for Educational Research, 2022).

There appear to be few studies that have investigated young non-dominant students' understanding of growing patterns, and little is known about Māori and Pacific students' representations of growing patterns and how they move through the stages of mathematical generalisation. The purpose of this study was to explore how drawing on culturally located tasks supported 10–12-year-old Māori and Pacific students to develop algebraic understandings and make sense of functional relationships. In particular, the following research questions were addressed: 1) What representations do Māori and Pacific students use when engaging with contextual functional tasks? 2) How do Māori and Pacific students generalise culturally located tasks involving functions?

LITERATURE REVIEW

Functional thinking entails generalising relationships between two or more varying quantities, representing and justifying these relationships in multiple ways, and reasoning with these generalised representations to understand and predict function behaviour (Blanton et al., 2015; Stephens et al., 2017). Functional thinking provides an important entry point for developing algebraic understanding and is regarded by many mathematicians as a powerful, unifying strand because functional thinking is threaded through all of mathematics, is a crucial part of mathematical development, and leads to a deeper understanding of the structural form and generality of mathematics (Kaput, 2017).

Research has found that young children are capable of deeper functional analysis than previously thought and that functional thinking begins at grades earlier than typically expected (Blanton & Kaput, 2011). Several researchers (e.g., Blanton et al., 2015; Stephens et al., 2017) have developed learning trajectories specifically for functional thinking, describing the typical progression of student thinking when generalising functional relationships. Three types of functional thinking are evident from the learning trajectories: recursive (looking for a relationship in a single sequence of values and indicating how to obtain a number in a sequence given the previous number or numbers), covariational (understanding how two quantities are coordinated and vary in relation to each other), and correspondence thinking (understanding the correspondence between the two variables and identifying an explicit rule so that a variable can be calculated from any term).

Researchers provide evidence that elementary school children can develop and use a variety of representational tools to help them reason with functions, describe recursive, covarying, and correspondence relationships, symbolise relationships, and express generalisations (Blanton et al., 2015; Cañadas et al., 2016). Representation is a dynamic process, and students move through different phases in their choice of representation, showing various levels of complexity in the ways they represent the relationships between two terms in a growing pattern (Blanton et al., 2015; Stephens et al., 2017). Representations explored in the literature include t-charts or function tables (Blanton & Kaput, 2011), visual (Moss & McNab, 2011), natural language (Radford, 2018), and symbolic representations (Stephens et al., 2015). Each representation provides a different way for students to examine and compare relationships, and students learn about and deepen their understanding of functions by exploring connections across multiple representations (Blanton et al., 2015; Cañadas et al., 2016; Stephens et al., 2017).

Mathematical tasks directly determine what learning opportunities are made available to students and are significant in establishing how students come to “view, develop, use, and make sense of mathematics” (Anthony & Walshaw, 2009, p. 13). Tasks embedded within cultural contexts provide important opportunities for non-dominant students to see themselves reflected in school mathematics and to recognise that the activities they engage in at home and in their communities involve mathematics, which is meaningful and valued (Hunter & Miller, 2022; Wager, 2012).

In the New Zealand context, there is some evidence that drawing on the mathematics implicit in Māori and Pacific patterning provides a powerful means of developing culturally diverse students’ early algebraic reasoning and understanding of functional patterns. Hunter and Miller’s (2022) research concentrated on the use of culturally located patterns from Pacific cultures to develop young students’ understanding of functional patterns and support generalisation. Their evidence showed that when mathematics was embedded in a cultural context, young, culturally diverse students were able to make a meaningful connection to mathematics, began to see covariation, developed their understanding of growing patterns, and articulated generalisations as they saw the structure of the pattern growing in multiple ways. Similarly, Miller and Warren's (2012) study investigated the role of culture in young Indigenous Australian students' mathematical generalisation of growing patterns. Results indicated that the type and context of the pattern impacted Indigenous students' abilities to access the structure and relationship between the variables. Young Indigenous students were more successful in extending and generalising growing patterns that came from the natural environment than patterns represented by decontextualised geometric shapes.

RESEARCH METHODS

The current research was conducted with a group of twelve students from one Year Six to Eight class (10–12-year-olds) in a low socio-economic, high-poverty, urban school in New Zealand. The ethnicities of the twelve students were Māori (40%) and Pacific (60%). Māori are indigenous to New Zealand and are a heterogeneous population composed of diverse groups and cultural identities (Greaves et al., 2015). Pacific peoples are a multifaceted population who originate or identify in terms of ancestry or heritage from the Pacific nations of Samoa, Cook Islands, Tonga, Niue, Fiji, Tokelau, Tuvalu, and Kiribati (Samu, 2015). Māori and Pacific peoples are distinct ethnic and cultural groups but share some cultural commonalities due to their geographical proximity and historical connections.

The current study drew on both qualitative case study and design-based research. Students engaged in an intervention of eight lessons focused on developing functional thinking. Tasks were designed to build on current student understandings and increase in complexity over a sequence of lessons. Each lesson began with the teacher launching the task with the whole group, followed by participants collaborating on the task in small groups. The teacher then facilitated a larger group discussion, connecting student approaches to key mathematical ideas.

The mathematics in all the tasks was embedded in a cultural context relevant to the student's backgrounds, with patterns drawn from Māori and Pacific cultures (see Table 1).

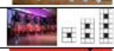
	Context	Pattern (visual)	Possible function
Task 1	Sāsā		$y=3x+1$
Task 2	Ngatu		$y=8x+4$
Task 3	Vaka		$y=6x-1$
Task 4	Titi		$y=6x+3$
Task 5	Tukutuku panel		$y=4x-6$
Task 6	Tivaevae		$y=24x+4$
Task 7	Fala		$y=x^2+x+4$
Task 8	Kapa haka		$y=\frac{x(x+1)}{2}$

Table 1: Cultural context, pattern, and function type of tasks used in the sequence of lessons

A range of data was collected and analyzed, including interviews, field notes, video-recorded classroom observations, and photographs of student work. Data was analyzed using thematic analysis to identify codes and generate themes to answer the research questions.

FINDINGS

Over the series of lessons, these Māori and Pacific students' learning progress mirrored what has been reported in research literature in relation to the typical trajectory of dominant groups of students' functional thinking (Blanton et al., 2015; Stephens et al., 2017). Shifts particularly occurred in how students i) used multiple representational forms to identify, communicate, and justify generalisations, and ii) expressed generalisations - from describing additive relationships using natural language to representing multiplicative relationships symbolically. The following sections of the paper describe the representations and generalisations these Māori and Pacific students used when engaging with culturally located tasks involving functions.

Using t-charts to support functional thinking

In the current study, the t-chart was the most commonly used representation to make sense of, explore, and generalise functional relationships across the series of tasks. In the early tasks, students created t-charts by making two columns of data and recording corresponding entries for the independent and dependent variables. The t-charts supported students in organising data and more easily noticing patterns as they mathematically explored the task's cultural contexts. All groups identified recursive patterns in the data by looking down the t-chart to find the additive difference between terms.

As the study progressed, students shifted from looking additively within columns to considering the horizontal relationship between the dependent and independent variables across the t-chart. Students began to use t-charts to think about the multiplicative relationship between the variables, realising that they could multiply the pattern number instead of relying on the recursive relationship. As one student explained: "Instead of plus six, plus six, plus six, you can times the pattern number by six".

Subsequently, the t-chart became a tool for identifying functional rules. Students worked out they could "find out what we're adding each time, multiply by that, then subtract something or add something". For instance, Figure 1 shows how a group used a three-column t-chart to find the additive difference (+6), then formulate the rule: multiply the pattern number by the common difference of six and add three.

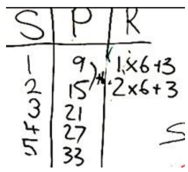


Figure 1: Using a t-chart to find an explicit rule

The final two tasks required students to use t-charts with an even greater degree of flexibility because the patterns were more complex quadratic relationships. The sequences were non-linear, so the strategy of looking for a common difference in order to formulate the multiplicative relationship between a number and its position could not be applied. The t-chart helped students identify an important property of quadratic functions: the second differences are the differences that have the common value (see Figure 2). For instance, this pattern has a second difference of 2, so it will be connected to the sequence of square numbers.

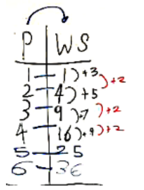


Figure 2: T-chart representing the quadratic pattern

These findings align with international research (Blanton & Kaput, 2011; Stephens et al., 2015), showing that as students' functional thinking developed, they transitioned from using the t-chart as an opaque object, a place to record numbers, to a transparent object, used to determine relationships in data, make explicit connections between variables, and derive function rules.

Visual representations to support generalisation

The visual representation of familiar cultural patterns was another effective means for non-dominant students to interpret and analyse functional relationships. Some students in the current study used drawing to explicitly identify the structure of the contextual patterns and make sense of the relationship between the variables. For example, Figure 3 shows how a group drew the eighth position to show the multiplicative structure of the growing ngatu pattern.

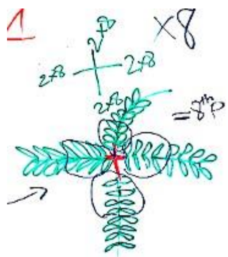


Figure 3: Visual representation of the multiplicative structure of the growing ngatu pattern

Engaging visually with the pattern gave students the opportunity to see its underpinning structure and connect the visual representation to multiplicative changes. As a result, this group was able to express a correspondence description of the relationship between the ngatu leaves and the pattern: "multiply the pattern by two, then multiply by four", or "you can just multiply the pattern number by eight".

Students also used visual representations of patterns to explain and justify their generalisations. For example, in the larger group discussion of the titi pattern, one group justified the rule they constructed using constant differencing from the t-chart with the way they drew the structure of the pattern. When students were probed for an explanation of the rule, they could show how the pattern grew in relation to the position number, how parts of the pattern changed, and how parts stayed the same, based on the visual configuration.

The ways these Māori and Pacific students used visual representations to develop pattern generalisation aligns with prior research (Moss & McNab, 2011), showing visual representations are a powerful means to support students to bridge the gap between the concrete situations represented in tasks and the abstract mathematical structures underlying the growing patterns.

Natural language to express generalisations

Tasks in the current study provided students with opportunities to use natural language around familiar cultural contexts to express generalisations. All the tasks integrated Māori and Pacific languages, and students were encouraged to draw on their home languages as part of the mathematical discussions.

Natural language was used by students in the first task to express factual reasoning as students attended to particular instances of the sāsā pattern. For example: "For the slaps, it adds on two. It goes plus two, plus two, plus two. It's two, four, six, eight, ten". Through natural language, these students indicated how to obtain the next number in the sequence given the previous number and expressed the recursive rule as an action within the pattern.

A contextual, natural language generalisation in the second task illustrated a shift in student reasoning from recursive to covariational thinking. Students coordinated the relationship between the leaves and the pattern position. For instance: "every time you move to the next pattern, the leaves grow by eight, and the stem stays at four". The language used revealed that the students' algebraic thinking was becoming more general.

In subsequent tasks, students conveyed correspondence thinking using more precise natural language. For example, in the tukutuku task, a group wrote: "if you times the rod with the crosses, then take away six that tells you how many crosses there are". This natural language generalisation provided evidence that these students were aware of the correlation between the two variables and could explicitly state a rule which described a generalised relationship.

Prior research (Blanton & Kaput, 2011; Cañadas et al., 2016; Moss & McNab, 2011; Radford, 2018) shows that as students' functional reasoning develops, so does their capacity to generalise growing patterns in natural language. The use of natural language is considered by researchers to have an important role in developing algebraic thinking because it allows learners to make sense of and describe algebraic concepts using language they know.

Symbolic representation of generalisations

In the current study, students spontaneously began to express function rules using a mixture of natural language and symbols as they started to look for more efficient ways to represent the generalisation. Early on in the study, a group proposed using symbols as a shorthand to represent the variables. "We started to shorten it by just writing the first letter to L, which is the L for leaves, and the first letter for pattern, which is P. So $8L \times P + 4$".

As problems increased in complexity, students continued to experiment with expressing the rule using variable notation. The sixth task was the first time that a group used symbolic representation before a natural language generalisation. In the final two tasks, there was evidence that symbolic thinking was beginning to lead student thinking: "The first rule we came up with was $P \times 2 - 1$".

Similar to the research (Radford, 2018; Stephens et al., 2015), the students had moved to a conceptual level of understanding where the natural language was receding into the background to make space for symbolic thinking and the more abstract signs of symbolic generalisations.

Connections between multiple representations

The use of multiple representations became common practice in both small group work and the larger group discussions. At the end of every lesson, the teacher ensured that all students were able to make sense of each other's explanations, make connections between the representations, and access progressively more sophisticated levels of representing and generalising functional relationships. Students recognised that when they used multiple representations to solve a task, they could check their reasoning because they arrived at the same conclusion in different ways. Additionally, student observations confirmed that they were able to triangulate natural language, numerical and visual strategies to justify their reasoning to peers. Each representation provided an alternative way for students to examine the structure of the pattern and the relationships between variables and highlighted different aspects of functional thinking (see Figure 4).

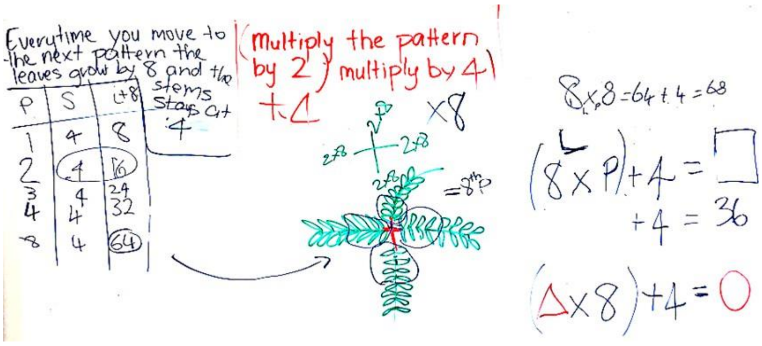


Figure 4: Multiple representations and generalisation of a growing ngatu pattern

This aligns with prior research (Blanton et al., 2015; Cañadas et al., 2016; Stephens et al., 2017) showing that students' flexibility with multiple representations promotes deeper mathematical insights, and students gain a more thorough understanding of functional relationships when they represent them in more than one way.

Cultural and mathematical identities

Data from the study revealed that the use of contextual tasks affirmed Māori and Pacific students' cultural identities. When students were asked how it felt to work on tasks related to their cultures, students expressed the idea that contextual tasks normalised their cultures in the mathematics classroom.: "It feels the same; I just feel like I'm Māori". Another student shared the feeling of belonging the contextual tasks engendered, stating: "It makes me feel welcomed. It makes me feel like I belong here".

Furthermore, when cultural contexts in the tasks were known and meaningful, students were positioned as experts and felt empowered as learners and doers of mathematics:

> We were solving a problem about a Samoan fala, and my parents are Samoan and so it made me feel like an expert. I never knew how to do algebra. I thought it was tricky as college problems, but the difficulty is alright, and it's fun. When it involves our culture once you hear it's something about your culture, you're

> like the expert because you know about a lot of things, and you're like, "oh yep, this is me, I know it", and then you just like relax and have fun while you solve the problem. It gives you confidence. And it gets your brain working.

Rather than perceiving mathematics at school as separate from their cultures, these students experienced strong cultural alignment with their Māori and Pacific identities and their mathematics class. These findings are consistent with previous national and international research (Hunter & Miller, 2022; Wager, 2012), showing that contextual tasks promote positive cultural identities by valuing the cultural capital that non-dominant students bring to the mathematics classroom.

CONCLUSION

There was significant growth in these Māori and Pacific students' conceptual understanding of growing patterns and in their ability to represent and generalise functional relationships in increasingly sophisticated and abstract ways. This contradicts non-dominant students' performance on national and international measures of mathematics achievement. The use of culturally located tasks acted as a lever for equity by blurring the line between cultural knowledge and content knowledge and provided opportunities for non-dominant students to engage in high-level algebraic reasoning and functional thinking. To challenge the gatekeeping role of algebra, it is important that educators provide opportunities for non-dominant students to draw on cultural resources in order to engage in high-level functional thinking and recognise that students' cultural knowledge and identities are assets that can propel them to success.

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TEACHING AND LEARNING EARLY ALGEBRAIC EQUATIONS IN A CHINESE CLASSROOM: A DESIGN-BASED RESEARCH STUDY GROUNDED BY THE THEORY OF OBJECTIFICATION

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The overwhelming presence of a procedural meaning of the equal sign and procedural understandings of equations reported in previous research calls for suitable interventions in teaching and learning algebraic equations. The aim of this research is to investigate the explicit features of interventions that address equation solving in Chinese early algebraic classrooms, and the emergence of procedural and relational understanding in children's learning process. In this study, a teaching-learning activity guided by the theory of objectification will be designed and implemented into Chinese elementary classroom, along with a Vygotskian multimodal semiotic analysis on the interactions between teacher and students. Note that his paper is a proposal of a future study along with some presentations of the pilot data.

INTRODUCTION

Children need to understand the equal sign (=), which symbolizes an equivalence relation between numbers or physical quantities, especially in learning arithmetic and early algebra. Their understanding of the equal sign may affect their arithmetic as well as linear equation solving performances (Jones et al., 2012). The relational understanding refers to viewing the equal sign as a relational sign that shows the value of both sides of the symbol are the same and able to define equality. However, among most western students, especially in the United States, the equal sign is treated as an operational symbol for "working out the answers," rather than a symbol of an equality relation. As a result, even with less explicit attention to the equal sign in higher grades, this misconception may still exist in middle grades, hindering their arithmetic and algebraic competence in future studies (Kunth, 2006). Although in China, the understanding of equations was focused on at the start of the fifth grade (e.g., see Li, 2008), a recent study by Xie and Cai (2022) showed that grade five students still tend to use a procedural strategy in solving equations, leading to a lack of understanding of equal signs, unknowns, and equation solving.

Previous studies have raised questions about what explicit activity should be considered as emergences of procedural or relational understanding of equal signs when interpreting and solving equations for Chinese students. Therefore, to investigate the emergence of equation understandings and suitable interventions, we have the following research questions:

1. How can the emergence of early algebraic thinking be identified in relation to understanding and solving algebraic equations?

2. What are the explicit features of instructional design that could support the development of early algebraic thinking concerning learning algebraic equations?

THEORETICAL FRAMEWORK

The theory of objectification is a mathematics educational theory developed by Luis Radford, inspired by the dialectical materialism of Marx and Hegel and the works of Vygotsky and Davydov. The theory of objectification considers mathematics education as a "political, societal, historical, and cultural endeavor" (Radford, 2021). Rather than simply looking at how well students perform in solving equations or correlating their understanding of equations with equation-solving performances, Radford takes a different approach by adopting a Vygotskian perspective of concept formation, which is based on the theory of objectification (Radford, 2021). This approach focuses on observing "the progressive development of conscious awareness of concepts and thought processes through interactions with teachers and peers" (Vygotsky, 1987, p.185), rather than just assessing final outcomes. When it comes to teaching and learning algebraic equations, Radford places particular emphasis on understanding the cultural and historical forms of concepts, i.e., the equal sign and equations. He also explores the underlying concepts that are required to solve equations, which emerge through interactions between teachers and peers (Radford, 2022). In the context of school learning, these interactions are achieved by teaching-learning activities. Indeed, teaching and learning are inseparable in the theory of objectification, which involves both knowing and becoming situated culturally-historically (Radford, 2021). Teaching-learning activity involves emotional, social, ethical, and intellectual factors and is a dynamic system formed by joint labour, which is the coordinated effort of individuals working toward a common goal. In this study, the teaching-learning activity involves both teachers and students working together to deal with algebraic equations. In this manner, teaching and learning are not two distinct activities but rather parts of the same process—the teaching-learning activity that relies on a cultural approach to solving equations.

Epistemologically, rather than the constructivists' view of knowledge as a product built up by cognizing objects and suggesting students construct their own knowledge and develop their intellectual autonomy, the theory of objectification defines knowledge as a "system of archetypes of thinking, action, and reflection constituted culturally and historically out of material, embodied, and sensible collective labour" (Radford, 2021). To derive the knowledge, it has to be put in motion and materialized to become concrete and sensible through activities and reflection, and such a way is defined as knowing (that shifts from concrete to abstract). Knowing in the theory of objectification aligns with Hegelians' view that knowing is to grasp the essence of objects through putting actions on those objects rather than approaching it through building our knowledge since "there is an entanglement between object and subject, world and consciousness" (Radford, 2021), and they co-produce and influence each other mutually.

METHODOLOGY

To design an intervention that is suitable for the Chinese classroom and investigate how students' learning process takes place in the intervention, an approach of design experiment (DE) would be suitable. For a general definition of DE, I would refer to Barab and Squire (2004), who describe it as "a series of approaches, with the intent of producing new theories, artifacts, and practices that account for and potentially impact learning and teaching in naturalistic settings." The preliminary goal of DE is to formulate a conjectured local instruction theory that can be examined and refined in classroom practice. From a research perspective, a theoretical intent need to be determined by setting an endpoint and a start point. The endpoint refers to clarifying mathematical learning goals that are determined by the cultural-historical context. The emphasis on establishing the most relevant goal guided by history and tradition is consonant with the theory of objectification, which requires that, prior to teaching-learning activity, an object of activity and goals should be defined. In terms of this study, the object can be the encounter of students with cultural-historical ways of understanding equal signs, and the goal would be to solve equations with formal or informal symbolism by holding a relational understanding of equal signs. Note that the aim is not to accept the existing school curriculum as it is and try to improve it, but to scrutinize the goals from a disciplinary perspective to establish the most pertinent and useful goals (Gravemeijer and Cobb, 2006). For starting points of local instruction theory, a psychological approach can be considered as documentations of the effect of instruction theory, while written tests or classroom observations can be used before initiating the experiment.

Implementation of design experiments involves a cyclic process of designing and testing instructional activities. In each cycle, the researcher conducts a preliminary thought experiment by envisioning how the proposed instructional experiments work in the classroom, and how students learn from anticipation (Gravemeijer and Cobb, 2006). In this study, data will be collected from a Chinese primary school, involving approximately 60 students. The classroom activity will be held in the form of a workshop for eight consecutive weeks, with 3-5 hours per week. The teaching-learning activity will be designed by the school mathematics teacher and the researcher through close collaboration. Since the learning process, or the process of materializing knowledge and instructions that support this materialization, will be the theoretical intent of this study, full classroom activities will be observed through video cameras. Two or more video cameras will be used to record both the teacher and students' activities and group discussions. Transcribed videos, students' written work, and field notes will be collected and used for data analysis. All data will be revised by both the researcher and schoolteacher during weekly research meetings. The purpose of revision is to make decisions about the validity of the conjectures that are embodied in the instructional activities. Here, I would associate this cycle of instructional activities with Simon's (1995) Mathematical teaching cycle. This concept suggests that the teacher and researcher should begin by predicting the mental activities when students engage in the planned instructional activity and determine the extent to which the observed learning process aligns with the initial predictions. Based on observation and revision, the teacher can consider potential or revised follow-up activities to better support the students' learning.

During the enactment of instruction, the interpretation of both students' learning and reasoning and means by which that learning is supported are key elements of design experiments which should be transferred into scientific interpretation through retrospective analysis (Gravemeijer & Cobb, 2006). The goal of retrospective analysis will depend on the theoretical intent of this study. Thus, I would conduct a multimodal semiotic analysis focusing on "talk, gesture, facial expression, body posture, drawing of symbols, manipulation of tools, pointing, pace, and gaze" (Nemirovsky et al., 2012) included in the data set of videos, field notes, and students' written works. A semiotic approach was used by Radford and Sabena (2015) to explore the phenomenon of teaching and learning that is modified within a cultural-historical context. To integrate semiotics into educational theory, Radford, and Sabena (2015) combined it with Leont'ev's (1978) Hegelian phenomenological concerns and

Vygotskian's cultural psychology and developed two methodological concepts related to the process of objectification in teaching-learning activity: semiotic nodes and semiotic contraction. Semiotic nodes refer to a joint activity in which various signs in the semiotic system are utilized together to achieve the objectification. This means that semiotic nodes represent mathematical interpretations and embodied action expressed by the teacher and students framed by cultural-historical forms of thinking and acting. Through identifying semiotic nodes, we can determine the mediation and materialization of knowledge, and where the objectification occurs. On the other hand, semiotic contraction refers to the reorganization and concentration of semiotic resources that happen as a result of students' increased consciousness of cultural-historical mathematics meaning. Students tend to make choices on what they perform depending on relevance with their awareness of underlying mathematical structures. As a result, they will refine the gestures they produce. Apart from semiotic nodes and contraction, the researcher will also examine students' written signs, oral descriptions, and embodied actions that relate to relational understanding of equal signs.

Data Collection on pilot data

In November of the previous year, I conducted a remote pilot study involving a collaborative teacher and five students at an educational institution in China. The instructional activities were devised by the researcher, and the collaborating teacher gave the tasks in the form of a workshop spanning a total of three hours, with one hour dedicated to each day. Throughout the implementation of the activities, the researcher observed that the collaborating teacher made modifications to the assigned tasks, leading to an incorporation of traditional Chinese education and the theory of objectification. The analysis will delve into both the adapted design by the collaborating teacher and the semiotic episodes within the classroom activities. This pilot study is considered the initial iteration of the upcoming study, contributing to the final result as a foundational component.

In summary the objectives of the pilot study include: 1. Ensuring that students comprehend the properties of equations using the balance, emphasizing concepts such as the equilibrium maintained when the same weight is added or removed from both sides. Students are then guided to simplify the balance by isolating the unknown quantity from the known ones and calculating the weight of objects. 2. students are tasked with observing equivalence relationships depicted by various balances and expressing them through both drawings and written descriptions, all while providing clear simplification steps.

The activity was divided into three sections: opening discussion, working in small group, closing discussion. In opening discussion, the teacher introduced the concrete pan-balance model, asking students whether they are familiar with how the pan-balance work, the teacher started with measuring weight of objects in classroom i.e., a cup or a watch. She put weight on the other side of pan to gradually make them balanced, and she start asking: "What does it mean when it is not balanced? And how about when it is balanced?"

For the material that constitute the main activity of teaching and learning algebraic equation, the concrete and iconic pan-balance scale is a specific manipulative to support students' understanding of relational meaning of equal signs i.e. Hiebert and Carpenter (1992) proposed that by associating the fulcrum of the balance with the equal sign, the pan-balance scale could serve as a model for developing a conceptual understanding of the equal sign, which means that the symbolic equal signs can be viewed as a denotation of quantitative sameness that aligns with the balancing model. This allows students to interpret arithmetic and algebraic problems in terms of quantitative symmetry or on the contrast, an unbalanced state, where there is a lack of symmetry between the two sides.

Here I would demonstrate two episodes of the classroom activity: The first one is solving equations via concrete pan-balance and the second one via iconic pan-balance. In the first episode, the collaborating teacher demonstrated the equation $x + 100 = 200$ on the pan-balance and ask students to solve the equation starting by isolating the unknown weight.

Teacher: This is 200 grams, the balance between them will happen soon…it is balanced now, the masses on both sides of my balance are equal, so my balance is in equilibrium on both sides. Now I want to isolate the known weight on the balance but retaining the balancing on both sides, how can I do this?

Wang: This is 200 (pointing the right side) …this is 100 and (pointing the left side) this is 200…

(Then Wang started removing one 100 grams on the right)

Teacher: Now, if you remove 100 grams from the right side of the balance, how much should you remove from the left side to balance the scale?

Wang: 100 grams. (Removing two 50 grams on the left)

Teacher: So, at this point, the masses on both sides of the balance are both 100 grams, and balance is restored.

In the first line, the teacher asked students to "isolate" the unknown weight, is designed to draw students' attention to the mathematical operations involved in simplifying the equation. "I want to isolate the unknown but retaining the balancing" have emphasized the equality should always hold when isolating the unknown. Also in Line 3, the teacher remind the Wang remove the weight on left side to retain the balance. When Wang starts isolating the unknown weight on left hand side in Line 2, the teacher named the operation by "removing". According to Vygotsky (1993), naming something not only makes the named entities more explicit in communication, but also elevates the prominence of the named entity, in essence, a word renders the named object more distinct in both consciousness and thought. Moreover, in Line 3, the teacher draws the attention of students on keeping the balance on both sides, leads to a relational meaning of equal sign when the concrete model transfers into symbolic ($x + 100 = 200$).

Next the teacher drew an iconic pan-balance on the board (see, figure 1), she asked: "We have a box of unknown apple and three apples on left side, and nine apples on the right side, each apple have the same weight, now I want to know how many apples in the box how can I know this?

Teacher: Think about the balance we played before, how to find the number of apples in the box?

Li: We need to put the box alone.

Teacher: Yes, put the box alone, and how can you do this? Remember we need to keep it balanced when you do that.

Li: (Came to the board and start crossing off the apples, but apple each from left to right) removing this and that… it is 6.

Teacher: Let's check if he is correct, did he isolate the box successfully?

Chen: Yes, there is only "X" on the left.

Teacher: Is the balance model retaining its balance? How can you know that?

Chen:	He (Li) crossed off 1,2,3…apples on left (pointing at the left), and 3 apples on the right (drawing a circle around the cross-offed apples), so it should be balanced.
Teacher:	That's right, if you cross off the same number of apples on both sides, the balance will be retained. We can substitute X equals 6 into the equation, where? On the left side. So, it's X plus 3 equals 6 plus 3, which equals 9 on the right side. Therefore, the equation holds.

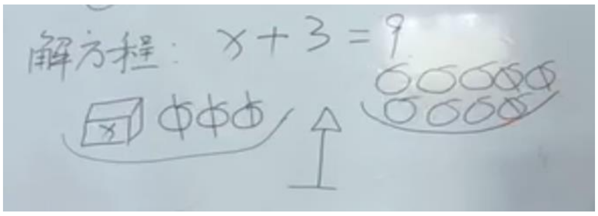

Figure 1. The iconic pan-balance model drew on the board, the Chinese word on left top means "solving the equation".

As above, the main idea for tackling this task involves isolating the box by subtracting an equal number of apples from both sides of the balance scale. This demonstrates that the balance remains unaffected. The process continues by gradually removing weights until only the weight of the unknown object remains on one side. In Line 4, Li isolates the box by crossing three apples on one side simultaneously without individually crossing each apple on a side. Li has effectively employed embodied action, treating the removed apples as a unified whole, which can be identified as a step of semiotic contraction (Radford & Sabena, 2015), this may also be found in the gesture of "circling" performed by Chen in Line 8. The response from Li, providing the answer of 6, further demonstrates that he understands the isolating process does not change the balance condition, even if it is not directly observable from drawing. This indicates that Li has grasped the properties of balance, specifically the concept of the equal sign, understanding that removing the same quantity from both sides does not affect the equality.

It is notable that after the teacher concluded the operations for solving equations, she provided various forms of equations involving addition and subtraction for students to practice in small groups. Then, the teacher wrote an expression in a general form, without specific numbers: "If we have $a = b$, think about on a balance, with a weight of a grams on the left and b grams on the right, now I want to remove or add the same weight, namely c grams on both sides, will it retain the balance? How can I exprees this in equation?" After peer discussion, she concludes the expression on the board: "If $a = b$, then $a \pm c = b \pm c$". Despite the teacher's inclination to use such an abstract form of expression, the researcher observed that the classroom activities followed a step-by-step progression, shifting from a concrete balance model to an abstract symbolic expression. Before the closing discussion, the teacher also introduced solving equations involving multiplication, including its algebraic form ($a = b \rightarrow a \times c = b \times c$).

DISCUSSION

Based on the general considerations of designing teaching-learning activities proposed by Radford (2018), some features of the designed activity were identified: 1. Devising an Artifact: The tasks were presented by introducing an artifact—the pan-balance—at the beginning of the instruction; 2. The tasks were given in an order of increasing complexity, from solving equations on pan-balance towards symbolic expression, from equations involving both numbers and symbols towards symbols only, referring to an increasing conceptual complexity; 3. The task were designed starting from a story problem that measuring weights or counting the quantity of apples, referring to a conceptual-contextual unity; 4. The teacher fostered collaboration among students through reflection and debate. For instance, in addition to small group discussions, the teacher encouraged students to justify the answers provided by their peers. This refers to the principle of fostering joint labor.

Here I would emphasize three levels of conceptualization in the classroom activity since they are crucial in the whole instructed activity. The first level involves a concrete sensual experience (Radford, 2021), at this level, students were able to visually perceive equality through concrete and sensory experiences via the pan-balance model. They observed that adding or removing weights could either alter or maintain the balance condition. The second level of conceptualization introduces theoretical reflection based on concrete objects, (Radford, 2021). In the classroom activity students manipulate the pan-balance by isolating the unknown weight both in concrete and iconic representation, opening up potential connections to the relational meaning equal signs through connecting the experience of the balancing condition and equality concepts. The third level of conceptualization emerges through the manipulation of mathematical symbols, enabling a transition from sensual experience to higher level of consciousness. In the classroom activity, the teacher encouraged students to represent the unknown weigh using their own expression (i.e., using "□" to represent unknowns or using arrows and crossing-off marks to represent subtraction), yet such experience was cut short when the teacher instructed them to substitute them into formal symbolism i.e., replacing the square "□" with the letter "x" The researcher suggests that introducing such symbolic representation should occur after students have engaged in symbol manipulation through the drawing of the balance, progressing from "removing" to "crossing" and finally to "subtracting".

Drawing on the issues introduced by the teacher regarding the generalized form of the relational property of equal signs, I would like to further investigate how students comprehend the concept in algebraic symbolic form in future studies. In response to the question of whether this form of expression is necessary for students, according to Radford (2003), although natural language enriches expressions related to designated objects, it becomes insufficient when there is a lack of words to designate the object. At this point, algebraic symbolism proceeds with a remarkable reduction. However, a more gradual process in which students progressively grasp the use of algebraic symbolism is preferred. I propose that by increasing conceptual complexity, i.e., through suitable conceptual variations (see Gu et al., 2017), as suggested by Radford (2021), the transition from embodied language to oral language and finally to algebraic symbolism is a crucial indicator of concept formation. A more sophisticated introduction of algebraic symbolism is recommended. It is noteworthy that students in this pilot study have comprehended the generalized algebraic expression of the relational property of equal signs, encompassing both additive and multiplicative structures ($a = b \rightarrow a \times c = b \times c$), Further investigation into students' awareness and understanding of these expressions is warranted.

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GENERALIZATION PROCESSES OF 5TH GRADE ELEMENTARY SCHOOL STUDENTS USING GRAPHS

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The literature recognizes the presence of algebraic structures in verbal and pictorial representations among elementary school students. The research objective is to describe the reasoning process employed by 5th-grade elementary school students (ages 10-11) when using graphical representations of functions to generalize. In this paper, we present the results of semi-structured interviews conducted with three of the children who participated in our study, wherein we presented them with contextualized generalization tasks involving functions. We give special attention to the structures observed during their work with function graphs and the process of reasoning they employ to reach generalizations. Our study leads us to the conclusion that engaging with the construction, reading, and interpretation of function graphs enables the participating students to identify structures within them.

INTRODUCTION

In the context of algebra, generalization is considered a fundamental concept, especially in the early years. This is because the introduction of algebraic thinking at a young age emphasizes the process of generalizing from the observation of regularities or patterns of behavior in a given mathematical situation (Radford, 2013). Among the various modes of algebraic thinking that address fundamental algebraic concepts in early grades, this research specifically focuses on functional thinking. Functional thinking centers on the relationship between two or more variables, encompassing various types of thinking that span from specific relationships to generalizations of these relationships (Kaput, 2008, p. 143). One of the key aspects of functional thinking involves generalizing relations between co-varying quantities. Also, it involves expressing these functional relationships using different representations and utilizing these expressions to analyze the behavior of a function (Blanton et al., 2011).

The reasoning process in this study focuses on the structures identified by students when they engage with generalization tasks involving linear functions. These tasks provide students with opportunities to explore and demonstrate their functional thinking. This aspect of their work has been recognized as a crucial area for further research (Stephens et al., 2017). We aim to address questions that remain unanswered in the existing literature, specifically regarding how young children generalize functional relationships between two quantities. More precisely, we seek to explore the types of relationships that children express and the levels of sophistication in their thinking about these relationships, which are still open questions (Blanton et al., 2015).

This paper aims to describe the process of generalization used by 5th-grade primary education students (ages 10-11) when working with graphical representations of functions.

THEORETICAL FRAMEWORK

The abductive-deductive reasoning model by Torres et al. (2021), based on Rivera's model (2017), establishes three phases of the reasoning process developed by students in a functional context (Figure

1). The abduction phase occurs when a conjecture is formulated based on limited specific cases, and the initial structures are identified. During this phase, preliminary hypotheses are generated but remain unconfirmed until a variety of specific cases are available. This is the point at which students need to identify a relationship between variables because, at this age, they lack tools for clear quantity visualization, counting, or drawing. Rivera (2017) refers to this as the generation of a hypothesis that will be confirmed in the subsequent induction phase. It is here that we observe whether the conjectures can be confirmed. A conjecture is confirmed when a student demonstrates the same structure on more than two occasions while working with distant specific cases. In this way, the student indicates an awareness of the implied regularity. The structure corresponds to how the elements of regularity between variables are organized and the relationship that exists between those elements (Kieran, 1989).

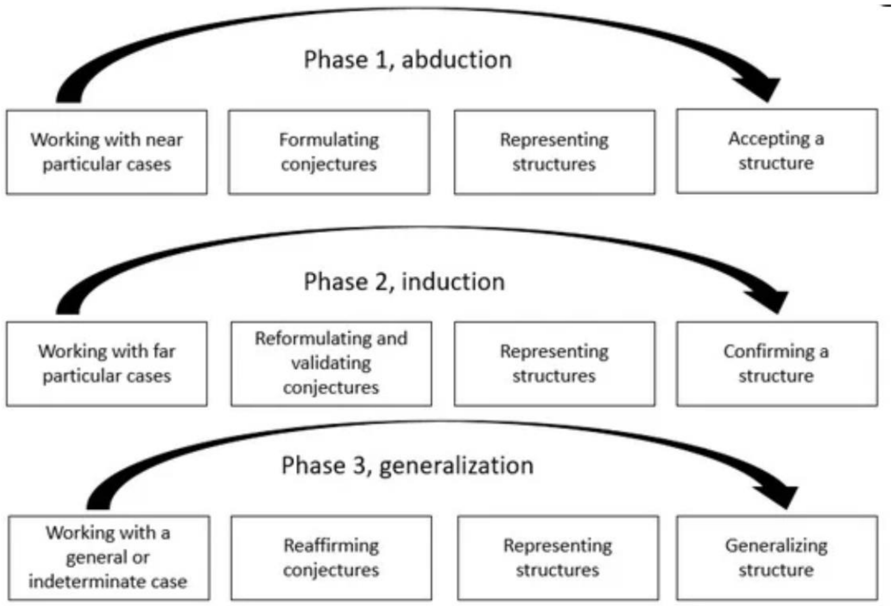

Figure 1. The abductive-deductive reasoning model by Torres et al. (2021), based on Rivera's model (2017).

From this moment onwards, it becomes possible to generalize through unspecified cases or the general case, as a tendency in the relationship between variables has been observed (Abe, 2003). Generalization involves establishing general relationships between covarying quantities, expressing these relationships through various representations (such as verbal, symbolic, tabular, and graphical), and reasoning with these representations to analyze the behavior of the function (Blanton et al., 2011).

METHODOLOGY

This work is part of a design-based research (Confrey, 2005) conducted with 25 students in the classroom, along with semi-structured individual interviews (Flick, 2012) conducted with six of these students. The students attend a public school located in the southern region of Spain. The only exposure this group had to graphical representation and generalization in functional contexts was through previous common sessions.

In this work, we analyze three interviews conducted by the researcher-interviewer. The task or stimulus focused on in this study is as follows: "You visit an amusement park where you have to pay 1 euro to get an entry card required for entry, and then, once inside, each ride costs you 3 euros." Students are tasked with calculating the total expenses based on the number of rides. The selection of interviews in this study is intentional, as each interview corresponds to one of the phases of the abductive-inductive model.

RESULTS

In the following section, we provide examples of student responses at each stage of the abductive-inductive model generalization process (Torres et al., 2021).

Abduction Stage:

This phase necessitates the formulation of conjectures through the exploration of potential structures while working with a limited number of specific and concrete cases. The representation of these structures holds significant importance during this stage since it serves as the medium through which initial conjectures are articulated. It is at this juncture that opportunities arise for their documentation. The abductive phase culminates with the acceptance of an a priori structure.

The ensuing dialogue showcases how Diane developed multiple conjectures while investigating potential structures within specific and concrete cases. When the researcher-interviewer inputted initial data into the graph, the student encountered a contradiction between the expected outcome (a multiple of three) and the structure she had initially posited. The graphical representation facilitated her discovery and acceptance of the structure; however, it did not yet empower her to generalize it or extend the graph to accommodate any value of the independent variable.

I: I took two trips and paid seven euros, aha. (The researcher-interviewer requests the data represented by Diane in the graph of figure 2).

D: But since seven is not a multiple of three, it couldn't be...

I: No?

D: Because when you divide seven by three, you get two, with one remaining.

I: And what do you think that remaining one represents?

D: That one... let me think... it could be from the entry card.

I: All right, now let me present a different scenario.

D: (Continues to ponder as the researcher prepares a new scenario and...) But in the graph, I didn't account for the entry card. In the graph, I directly input... (picks up the graph again) Let me include the entry card. (Makes some modifications but realizes that the graph remains the same as before). I had it right.

I: So, you were correct, weren't you?

D: Yes

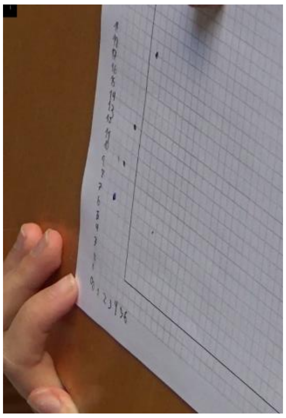

Figure 2. Initial Graph of Diane. Abduction Phase

Diane has two conjectures for the hypothetical relationship between variables. One she initially represented in the graph (f(x) = 3x+1) and the other arises when asked by the researcher-interviewer (f(x) = 3x). As Rivera (2017) points out in his research, the student is generating a conjecture that is not yet definitive.

The student represents well in the graph the functional relationship because she thinks in context. When she moves away from the context, she begins to doubt her generalization conjecture:

D: But since seven is not a multiple of three, it couldn't be...

I: No?

D: Because seven divided by three is two with one left over.

And she returns to the functional relationship when the researcher-interviewer brings her back to the context:

I: And that one, what do you think it's from?

D: That one... would be... (thinks) from the entry card.

The graph in Figure 3 shows the final result after the doubts expressed in the functional relationship. It can be seen that the initial conjecture remains although there are signs of her insecurity in the erased points. It is typical of this phase to represent only a small group of points with which she has tried to identify the possible initial relationship.

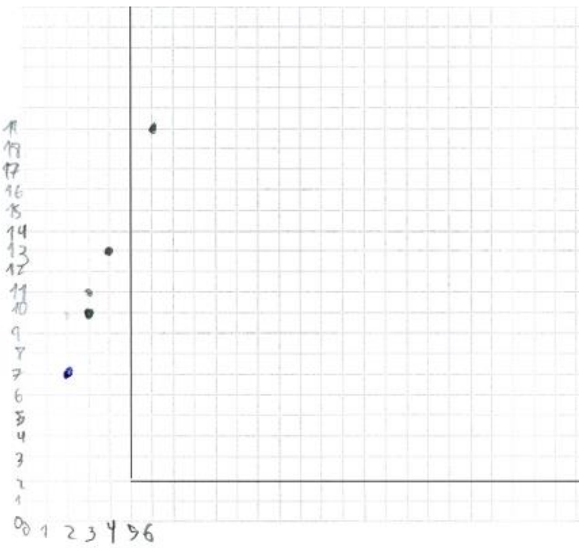

Figure 3. Final Graph of Diane

Induction Phase:

Meli represents the situation of the amusement park in the graph of Figure 4. This graph does not contain structural elements as defined by Martí (2010). It lacks labels on the axes, nor does it have numbers on the axes (only the first three are shown), but she can construct the graph because she has identified the rate of change between the variables.

I: How are you plotting the points?

M: I'm plotting it for each ticket because it costs 3 and I've already paid the euro for the ID card at the beginning...

In the case of Meli, the context of the task plays a fundamental role in the graphical representation of the functional relationship.

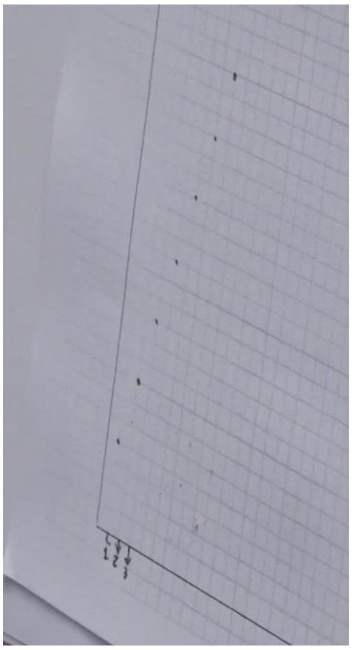

Figure 4. Grph of Meli. Induction Phase

During this phase, Meli works with a broader range of specific cases. In the dialogue, he revises his conjectures, pinpointing an additional structure that wasn't apparent during the abductive phase. The distinctive aspect of this phase is the need to validate the viability of the structure expressed while working with numerous individual cases.

M: Looking at the graph... if this one is here (points to the point corresponding to x=2), here (moves up three units and one to the right, placing the correct point).

I: Okay, what if there were four?

M: (attempts to replicate the same action but realizes it doesn't align with the next point, the one previously placed for x=5, which resulted in y=16 - a clear mistake) No, because this is not three, but four.

I: Four? Let's check if this one isn't correct...

M: (Realizes the error, reaches for the eraser, and removes the incorrect point) That was my mistake...

I: You see, because what you're explaining to me is that they occur...

M: in threes.

In this presented dialogue, Meli identified a structure within the points she had graphed previously. Initially, she encountered contradictions between the arithmetic structure she used to plot the dots on the graph and the structure that the dots on the graph adhered to.

Generalization Phase:

It is at this juncture that we can generalize based on either the indeterminate cases or the general case, as a discernible pattern has been observed in the relationship between the variables (Abe, 2003).

The following dialogue is an example of why we place Martín in this phase.

I: (The researcher-interviewer encourages Martín to construct a graph of the situation and explain what he is doing)

A1: And here I could mark it here (starts putting points on the graph, puts the first two based on the data from a previously constructed table), or I could also do... (and starts writing all the points following the trend shown in the graph)

In the continuation of the previous episode, Martin generalizes the relationship between the variables and represents them on the graph by following the rate of change between them.

I: How are you constructing that?

MA: How? Well, I observe... First, I work with the initial two or three data points, and I ponder how it progresses. As I've come to realize that it advances by three units horizontally and one unit vertically, I continue to plot additional points in this manner (proceeding to add more points where there are no numerical values on the axes, aligning with the observed trend).

The graph constructed by Martín is shown in Figure 5. Unlike Diane, who was in the abduction phase, Martín continues to represent points regardless of their number. He could have continued representing points because he is confident in knowing the pattern they follow on the graph.

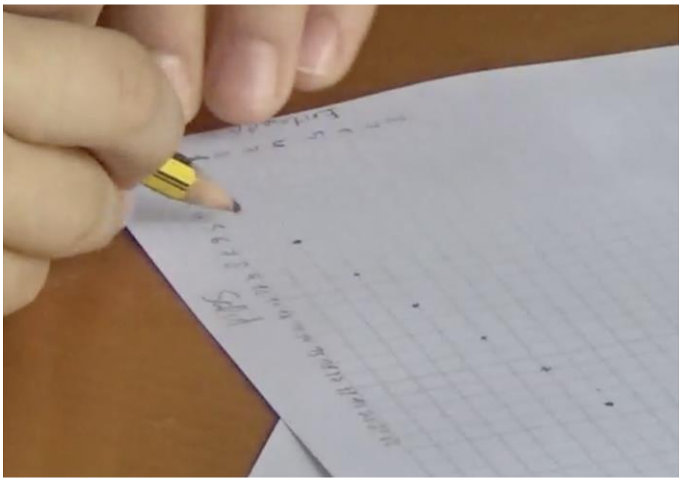

Figure 5. Graph of Martín. Generalization

CONCLUSIONS

The research literature identifies structures represented both algebraically and pictorially during the generalization process. The research results demonstrate that constructing, reading, and interpreting function graphs enable primary school student participants to discern structures within function graphs as well. These graphical structures empower students to position themselves within various phases of the abductive-inductive cycle to generalize the functional relationship and represent it on a function graph.

The analyzed examples suggest that the phases of the abductive-inductive reasoning model (Rivera, 2017; Torres et al., 2021) can be recognized in working with graphs. In the abduction phase, the graph contains few points, and different relationships between the variables are tested. Its construction employs the possible functional relationship to calculate the coordinates of each point. The context is taken into account to locate each point.

In the induction phase, they test their possible relationship between variables with more points on the graph. They begin to identify common characteristics in the arrangement of points on the graph.

Generalization involves identifying the relationship between variables. They recognize the trend that the points follow and use it to build and extend the graph. They reason with the graphical representation and employ the structural elements of the graph for this purpose.

The analyzed responses indicate that the context of the task allows students to construct the initial points of the graph, but once they generalize, the construction of the graph is carried out without considering the context. Instead, they use the regularities detected between the variables and the way they are reflected in the graph.

Acknowledgment

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COMMON ERROR IN THE RELATIONSHIP BETWEEN PARTS AND WHOLE IN LOWER ELEMENTARY SCHOOL GRADES

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In this study, we created an evaluation task for algebraic thinking, focusing on the "relationship between parts and whole" with four components: commutativity, equivalence, covariance, and completeness. We conducted a cognitive survey on early elementary school students, revealing that their understanding of this relationship improved significantly from first to second grade. However, they struggled with understanding covariance and completeness, often giving "insufficient" or "unnecessary" answers due to difficulty comprehending structural relationships in instructional text.

BACKGROUND AND PURPOSE OF RESEARCH

Numerical development is linked to counting principles and strategies (Fuson, 1982; Gelman & Gallistel, 1978). Recently, a structural approach focuses on understanding numbers through parts and wholes, with demonstrated effectiveness (Björklund et al., 2018; Kullberg et al., 2020). However, clear assessment tasks for this relationship are limited, especially in lower elementary grades. This study aims to define the elements of the 'relationship between parts and wholes' through a literature review and create assessment tasks for use in lower elementary grades.

Therefore, based on the organization of previous studies (Ekdahl et al., 2016; Falkner et al., 1999; Kullberg et al., 2020), we tentatively set four components regarding the "relationship between parts and whole": commutativity, equivalence, covariance, and completeness.

This study has two goals: first, to define the elements of 'the relationship between parts and wholes' through a literature review and create assessment tasks; and, second, to use these tasks in lower elementary grades to understand the nature of incorrect responses.

METHOD

Survey Period and Participants

This survey was conducted in November 2022. The participants in this survey were a total of 392 students from two public elementary schools in H Prefecture, comprising 204 first-graders and 184 second-graders. The response rates for each grade were 112 students (54.9%) for the first grade and 132 students (71.7%) for the second grade. It should be noted that in both elementary schools, students had previously learned 'number composition and decomposition,' addition with carrying, and subtraction with borrowing.

Survey Method

In this survey, we distributed survey forms and solicited free responses. The response time was approximately 45 minutes, and after completion, we requested voluntary submissions.

Survey tasks

In this survey, following previous research, we tentatively identified four components of understanding the "relationship between parts and the whole": commutativity, equivalence, covariance, and completeness. Each perspective had four corresponding questions, as detailed in Table 1 below.

Components	Instructional text and Numerical values
Commutativity	①This 4 candy can be divided into 3 pieces and 1 piece. Choose one similar way to divide the candy and circle it. (1) 1 and 3 (2) 2 and 2 (3) 0 and 4 (4) 2 and 3 (5) 2 and 1 (6) 4 and 0
	②5(2,3), ③ 7(2,5),④ 8(6,2) (The number on the left indicates the whole number, and the number in parentheses indicates the number of parts to be presented.)
Equivalence	①Choose three ways to make four pieces of chocolates in all. (1) 1 and 3 (2) 2 and 2 (3) 4 and 1 (4) 2 and 3 (5) 3 and 2 (6) 0 and 4
	② 5 (2,3)(4,1)(5,0), ③ 7(2,5)(4,3)(1,6), ④ 8(2,6)(3,5)(1,7)
Covariance	①Divide 3 of the 4 cookies into round dishes and 1 into square dishes. How many cookies will be in a square dish if you move 1 from a round dish to a square dish?
	②5(3,2)[2], ③ 7(5,2)[1], ④ 9(3,6)[2 (The number on the left is the whole number, the number in parentheses is the number of parts to be presented, and the number in [ ] is the number to be moved.)
Completeness	Divide the ball into two groups. Draw all the combinations of the numbers that fit in the boxes. ① The 3 balls are □ and □.
	②4， ③7， ④8

Table 1: Instructional text and numerical values for each problem

Analysis Method - Classification Method

First, each task was categorized into correct and incorrect responses. Incorrect responses for each task were further classified as follows, including "no answer" (where no response was provided) and "unknown" (where the intention was unclear) in the classification of incorrect responses.

Commutativity: The task was classified into "whole" and "one side." "Whole" is a response that lists various ways to divide it where the whole number is equal. "One side" is a response that selects a way to divide it where one side of the given parts is equal.

Equivalence: The responses were categorized as "insufficient" and "unnecessary." "Insufficient" refers to a response in which only two or only one of the three correct answer choices was selected. However, unnecessary choices were not selected. "Unnecessary" indicates that the respondent selected the correct answer choice but also chose an unnecessary option.

Covariance: "The "wrong target" and "inverse operation" were categorized. "Wrong target" is a response in which the target of the change in the number of two parts is incorrect. "Inverse operations" are those in which the answer is obtained by subtraction instead of addition.

Completeness: The task, similar to the equivalence task, was categorized into "insufficient" and "unnecessary." Additionally, responses that equated the commutative property and listed only one of them were classified as "commutative."

RESULTS

Regarding the understanding of the "relationship between parts and whole," Table 2 presents the correct and incorrect response rates for each grade and element. After scoring, a two-factor mixed-design analysis of variance was conducted, considering the grade factor and element factor. As a result, the main effect of the grade factor ($F(1, 242) = 8.08$, $p < .05$, $\eta p2 = .032$) and the main effect of the perspective factor ($F(3, 726) = 109.92$, $p < .01$, $\eta p2 = .312$) were significant. Additionally, the interaction ($F(3, 726) = 0.54$, $p = .62$, $\eta p2 = .002$) was not significant. Therefore, concerning the perspective factor, post hoc multiple comparisons were conducted using Tukey's HSD method, revealing significant differences among all elements.

	1st grade (n=112)					2nd grade (n=132)			
	Commutativity	Equivalence	Covariance	Completeness		Commutativity	Equivalence	Covariance	Completeness
corrct	70.5%	77.7%	52.2%	13.6%		84.5%	86.9%	60.4%	15.9%
incorrect	29.5%	22.3%	47.8%	86.4%		15.5%	13.1%	39.6%	84.1%

Table 2: " Parts and Whole task” correct and incorrect rates for each grade level.

Next, as shown in Table 3, the results of error analysis are presented. In the commutativity task, over half of the responses were categorized as "whole." The next most common error was "no response," accounting for one-third of the errors. In the equivalence task, errors in selecting both correct choices were very few, while responses with only one selection or unnecessary choices together accounted for 40% to 50% of the errors. Other responses were mostly "no response." In the covariance task, significant errors could not be discerned, and "unclear" responses constituted 50% to 60%. The most frequent error among errors was "wrong target," making up just under 20%. Additionally, in the first grade, "inverse operation" was observed in about 10% of cases. In the Completeness task, the most common error was "insufficient (1)," occurring in 40% to 60% of responses. Others included "insufficient" where some correct answers were missing and "no answer".

DISCUSSIONS

Based on error analysis, in the commutativity task, over half of the responses were categorized as "whole," likely because participants misinterpreted the instructional text's implied "similar ways of grouping." This corresponds to the concept of equivalence. In the equivalence task, very few errors occurred when selecting both correct choices, and it is assumed that when participants understood the concept of selecting multiple options, they mostly answered correctly. However, in cases where they did not grasp the concept of selecting multiple options, they likely chose only one. Furthermore, when they couldn't comprehend the concept of "making all the numbers the same," it often resulted in no response. In the covariance task, identifying errors was challenging, with "unknown" responses at 50-60%, highlighting survey limitations and the need for qualitative investigations. In the completeness task, most responses involved selecting only one option, indicating a lack of understanding of the directive to "write down all combinations." Participants could decompose numbers but lacked structural understanding for other combinations.

	1st grade (n=112)		2nd grade (n=132)	
	Average response	incorrect rate	Average response	incorrect rate
Commutativity				
Whole	18.5	56.1%	12.75	62.0%
One side	2.5	7.6%	0.5	2.4%
No answer	12	36.3%	7.25	35.6%
Unknown	0	0.0%	0	0.0%
Equivalence				
Insufficient(2)	1.25	4.9%	0	0.0%
Insufficient(1)	5.5	22.1%	5.75	33.2%
Unnecesary	5.75	23.1%	3.75	21.5%
No answer	11.5	45.9%	6.5	37.9%
Unknown	1	4.0%	1.25	7.4%
Covariance				
Wrong target	7.5	14.4%	9.75	18.7%
Inverse opreration	7.25	13.6%	2	3.8%
All-sum	1.5	2.7%	2	3.8%
No answer	11	20.5%	5.5	10.5%
Unknown	26.25	48.8%	33	63.2%
Completeness				
Insufficient	16.25	15.8%	16.5	14.2%
Commutative	2.75	2.6%	1.75	1.5%
Insufficient(1)	39.75	42.0%	66.75	60.9%
Unnecesary	3.75	4.0%	3	2.7%
No answer	23.25	24.0%	13.25	11.9%
Unknown	11	11.6%	9.75	8.8%

* "Average responses" is the average number of responses for each category.
* "Incorrect rate" is the rate in each category per number of incorrect responses in each question.

Table 3: Error analysis of each task.

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STUDENTS' CONSTRUCTION AND REORGANIZATION OF MEANINGS ABOUT MULTIVARIATION

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This paper describes how sixth-grade students reasoned multivariationally as they manipulated variables while interacting with digital simulations of clouds, the rock cycle, and a hot air balloon. We provide data to illustrate three different mental processes that students used to construct and reorganize their meanings of multivariation, namely bridging, transforming, and reforming. Our findings initiate a discussion on how multivariational reasoning can become accessible to a younger population of students.

BACKGROUND

Most real-world phenomena involve complex relationships of multiple variables changing simultaneously that people are expected to examine and understand. Despite this societal need, the mathematics teaching at school usually restricts students' explorations to phenomena studying change in one (variation) or two variables (covariation) only. Likewise, extensive research in math education has focused on characterizing and supporting students' variational and covariational reasoning (e.g., Carlson et al. 2002; Ellis, 2011; Thompson & Carlson, 2017), where the latter involves envisioning two quantities' values varying simultaneously. Quantities are measurable conceptual attributes constructed by individuals as they conceive an object's quality (Thompson, 1994).

An exception to this lack of multivariation research is Jones' (2018) work in undergraduate education who used the context of differential equations to characterize students' forms of multivariational reasoning based on the relationship they constructed. Jones refers to *independent multivariation* as a situation where some quantities can be held constant while others vary, while in situations of *dependent multivariation* a change in any one quantity produces simultaneous changes in all other quantities. In *nested multivariation* the quantities are related in a function composition structure $z(y(x))$ where a change in x has a corresponding change in y, and that change in y then corresponds to a change in z.

In our most recent work (Panorkou & Germia, 2023) as part of a project studying math and science interdisciplinary learning, we studied sixth-grade students' multivariational reasoning and compared those to Jones' (2018) characterizations with undergraduate students. Among other forms, we introduced *partial dependent multivariational reasoning* in which students reason about how an independent quantity influences the simultaneous change of two dependent quantities that are not related to each other, and *integrated multivariational reasoning* in which students merge more than one form of multivariational reasoning in their statements (such as using both independent and nested multivariational forms). That study illustrated the need to investigate how these different forms of reasoning are constructed and reorganized in order to understand how to best support students' multivariational reasoning. Consequently, in the current study we examined how students construct meanings about relationships of multiple quantities and especially how they reorganize meanings from earlier forms, such as covariation. Specifically, we investigated, how are students' multivariational meanings constructed and reorganized from earlier forms of meanings?

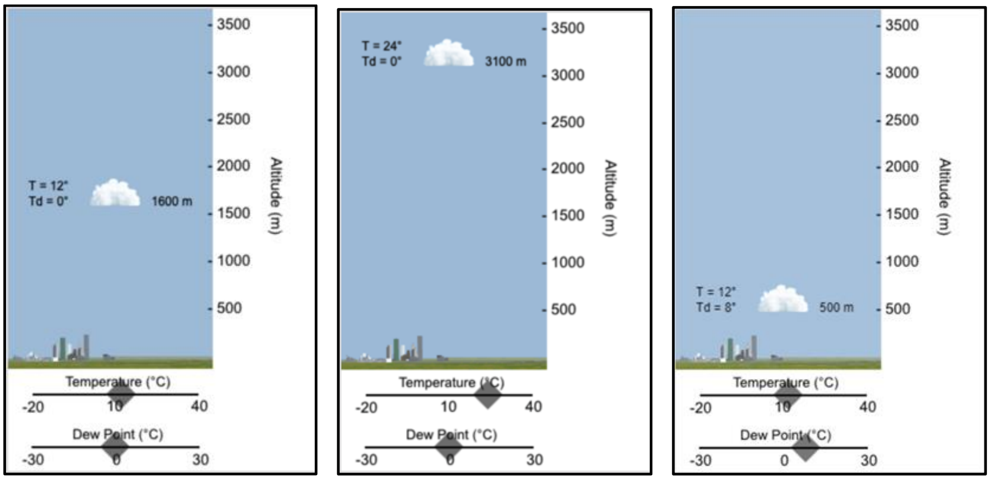
We characterized each meaning as a scheme of meanings and ways of thinking that builds on students' prior meanings of quantities and incorporates more sophisticated meanings than before. By reorganization (Piaget, 2001), we refer to humble inferences we make about their reflections and projections of particular meanings about the quantities and their relationships to a higher conceptual level where these initial meanings become part of a more coherent whole. We aimed to understand how students' meanings about varying quantities could be shaped and reorganized as students interact with our task design, simulations, and questioning.

DESIGN AND METHODS

In this paper, we present our analysis from two whole-class design experiments (Cobb et al., 2003) in two sixth-grade classrooms from the Northeast of the United States. In one classroom, students explored the phenomenon of weather by interacting with digital simulations focusing on the height of clouds and the hot air balloon. The second classroom explored the phenomenon of rock cycle by interacting with the Bob's Life simulation. Students were asked to freely explore the simulations and then explain the relationships they notice.

The "How high are those cumulus clouds???" Simulation

The "How high are those cumulus clouds???" simulation (Whittaker, 2015) displays the changes in the altitude at which a cumulus cloud forms according to changes in the ground temperature and the dew point temperature. The user can drag the two sliders, one for the ground temperature (T) and the other for the dew point temperature (Td), to see the corresponding changes in the altitude at which the cloud forms (Figure 1a). For example, when the ground temperature changes from 12 °C to 24 °C while the dew point temperature remains set at 0 °C, the altitude of the cloud changes from 1,600 m to 3,100 m (Figure 1b). Similarly, when the dew point temperature changes from 0 °C to 8 °C while the ground temperature remains set at 12 °C, the cloud's altitude changes from 1,600 m to 500 m (Figure 1c).

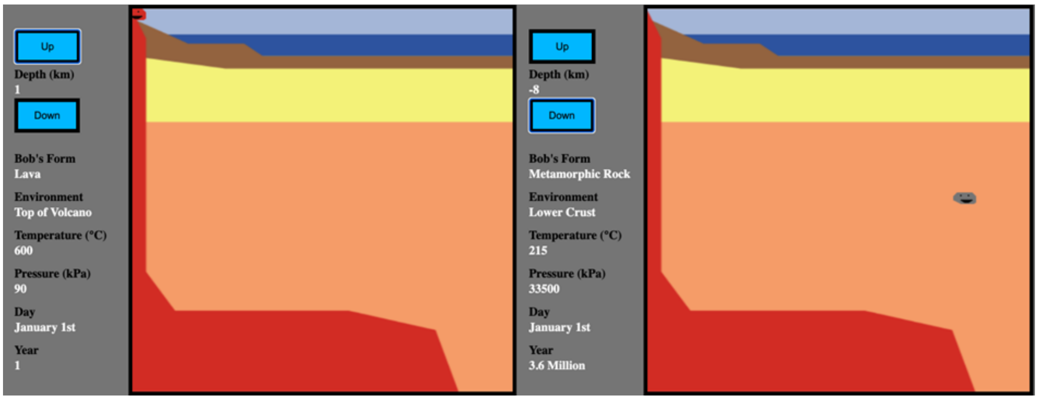


(a) (b) (c)

Figure 1: The "How high are those cumulus clouds???" simulation

Bob's Life Simulation

We designed the Bob's Life simulation to show one possible path of a rock named Bob traveling through the rock cycle. Bob starts at the top of a volcano as lava (Figure 2a). The student can change Bob's depth in kilometers using the "up" button (decreasing the depth) and "down" button (increasing the depth). Changes in Bob's depth result in changes in Bob's color, form, environment, temperature, pressure, and the time since the volcano erupted. For instance, when Bob is 4 km below sea level (-4 km), Bob is a sedimentary rock found in the earth's upper crust at a temperature of 40 °C and a pressure of 4,000 kPa. But when Bob is 8 km below sea level (-8 km), Bob is a metamorphic rock found in the earth's lower crust at a temperature of 215 °C and a pressure of 33,500 kPa (Figure 2b).



(a) (b)

Figure 2: The Bob's Life Simulation

The Hot Air Balloon Simulation

The Hot Air Balloon simulation shows the relationship between the size of the flame in the balloon, the temperature of the air inside the balloon, the density of that air, and the balloon's altitude. The student can change the temperature of the air inside the balloon using the "turn flame up" and "turn flame down" buttons. Increasing the size of the flame also increases the temperature of the air inside the balloon, which decreases the density of that air, which increases the balloon's altitude (Figure 3).

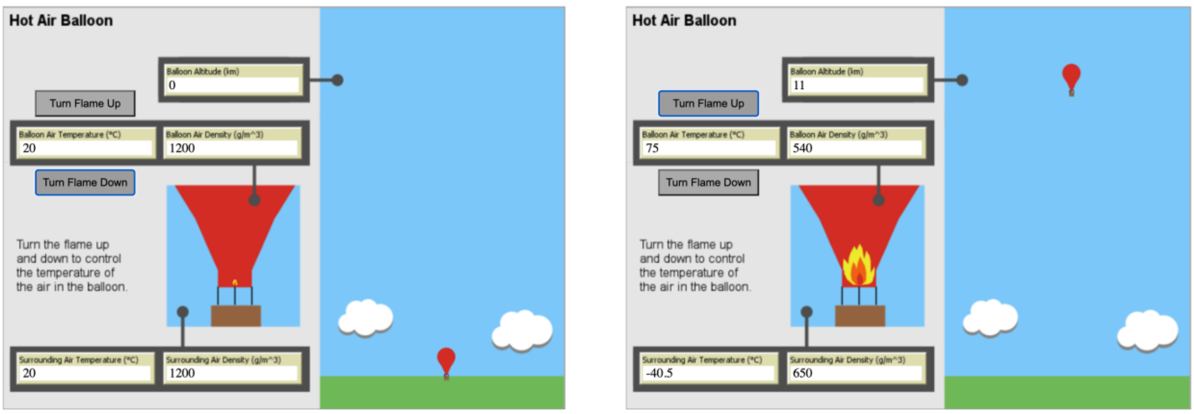


Figure 3: The Hot Air Balloon simulation

DATA COLLECTION AND ANALYSIS

Due to COVID-19 data were collected virtually on Google Meet. The design experiments sessions ranged from 15 to 30 minutes. In each class, we worked with pairs or small groups of students to create a small-scale version of a learning ecology in order to study it in depth and detail (Cobb et al., 2003). Both the whole class and small group interactions were video- and audio-recorded. At the end of the sessions, we conducted a retrospective analysis (Cobb et al., 2003) to identify student excerpts that exhibit variational, covariational, or multivariational reasoning as described in our framework. The data were independently coded by three researchers. Any coding disagreements were fully resolved after discussion. In this paper, we present the retrospective analysis (Cobb et al., 2003) of two pairs of students to discuss the constructions and reorganizations of their meanings in depth as they engaged with the two simulations. Anne and Violet worked with the weather simulations while Billie and Aidan worked with the rock cycle simulation.

FINDINGS

Here, we discuss the two pairs' constructions and reorganizations of reasoning about quantities as they engaged with the three simulations.

The "How high are those cumulus clouds???" - Anne and Violet

When we asked students to explain what the simulation told them, Violet replied that it showed "what happened if the dew point gets lower, it [the cloud] rises." She later added that "when you put the [ground] temperature higher, the altitude increases." These observations show that Violet was making connections in pairs of quantities, specifically the dew point or the ground temperature with the cloud's altitude, thus reasoning covariationally (Table 1). Anne engaged in a similar reorganization, stating that "if you move the [ground] temperature up, it [the cloud] goes high." She later stated a similar description of the relationship between the dew point and the cloud's altitude. Violet and Anne's reasoning included considering the direction of change of these quantities.

We next encouraged students to describe how the altitude would change if they changed the two temperatures, ground and dew point. In response, Violet stated that "when you put the [ground] temperature up, it changes the altitude, but when you put the dew point up, it [the altitude] gets lower." We interpret Violet's statement to show that she bridged the two covariational relationships to reason multivariationally about how changes in both the ground temperature and the dew point independently caused different changes in the cloud's altitude at the same time. We then asked Anne to state this relationship in her own words. She reasoned:

> When you keep the ground level, like the dew point to zero, if you … make the number [ground temperature] go higher with the cloud, the numbers on this side of the cloud [altitude], they start to go up. And … if you move the [ground] temperature to a lower number … the numbers on the side of the cloud [altitude], they start to go down also.

In this excerpt, Anne showed that she was reasoning multivariationally about all three quantities being related to each other. In contrast to Violet, however, Anne did not appear to envision all three quantities to be changing at the same time. Instead, she kept "the dew point to zero" to fix that quantity and then reasoned about changes in the other two quantities in that specific situation. This is an example of viewing a quantity, in this case dew point, as a parameter (Thompson & Carlson, 2017) which can change but does not do so within a given situation.

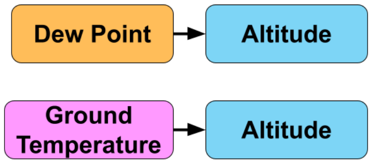
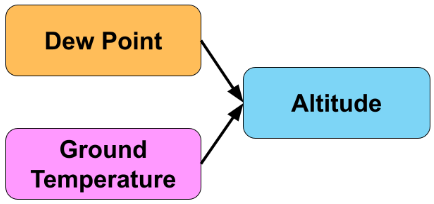
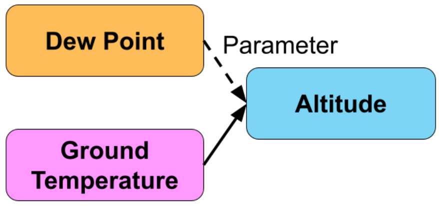
Reorganization	Initial Relationship(s)	New Relationship
Bridging		Violet  Anne 

Table 1: Anne and Violet's Forms of Reorganizations of Relationships in "How high are those cumulus clouds???"

The Bob's Life (Rock Cycle) - Billie and Aiden

When we asked them to describe the relationship between Bob's depth and temperature, Aiden observed that "Bob's temperature goes higher when Bob's depth goes lower." Similarly, Billie stated that "every time the rock gets deeper, the numbers, the numbers get lower. And then the temperature gets higher." We interpret these excerpts to show that Aiden and Billie reasoned about these quantities covariationally by envisioning the changes occurring in both as happening at the same time (Table 2). They also gave attention to the direction of change of these quantities, as evidenced by their use of words such as "higher" and "lower" in their statements. When asked a similar question about depth and pressure, Aiden exhibited the same reorganization of his reasoning again and replied, "While Bob gets more deep, the pressure goes higher. He gets more pressure."

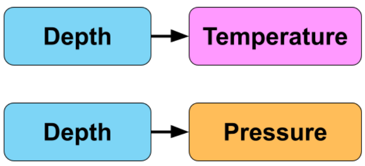
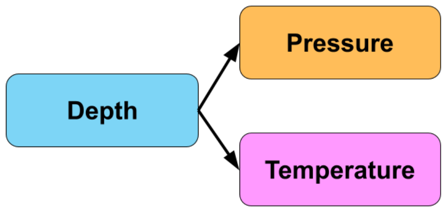
Reorganization	Initial Relationship(s)	New Relationship
Bridging		

Table 2: Billie & Aiden's Forms of Reorganizations of Relationships in Bob's Life

To encourage students to merge the two relationships they had reasoned about, depth-temperature and depth-pressure, we then asked about the relationship between all three variables. Billie answered, "The temperature and pressure get higher, meanwhile, the depth gets lower." Similarly, Aiden observed that "the deeper it gets, the more pressure and temperature it gets." Like Anne and Violet, we interpret these statements to show that both students were able to bridge their covariational reasoning into a multivariational envisioning of all three quantities changing at the same time. In contrast to Anne and

Violet's independent multivariational reasoning, Billie and Aiden exhibited partial dependent multivariational reasoning as they reasoned about how the depth (an independent quantity) influences the simultaneous change of temperature and pressure (dependent quantities) that are not related to each other.

The Hot Air Balloon - Anne and Violet

To reason multivariationally, Anne and Violet exhibited three forms of reorganizations of their earlier meanings. Table 3 provides an example of each form.

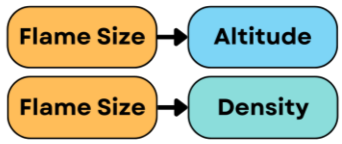
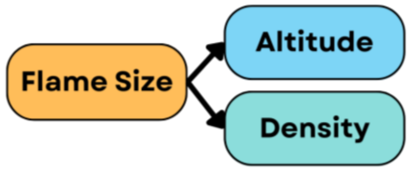
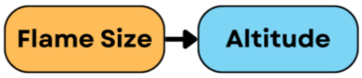
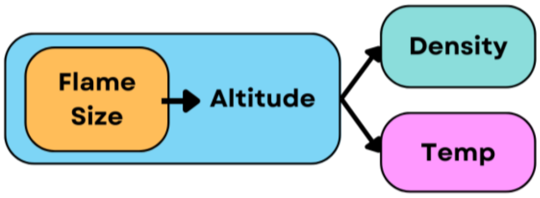
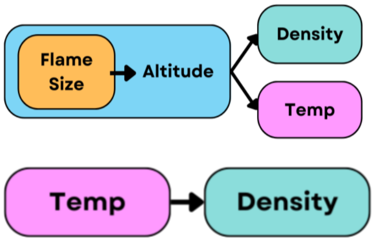
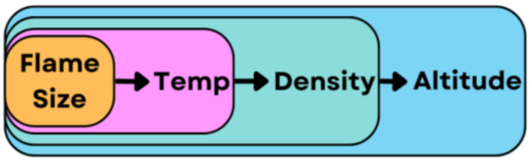
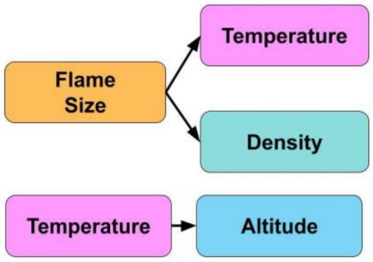
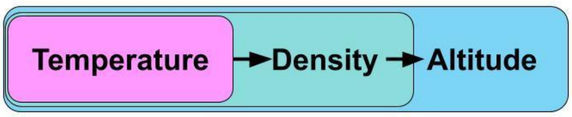
<table>
<tr><th>Reorganization</th><th></th><th>Initial Relationship(s)</th><th>New Relationship</th></tr>
<tr><td>Bridging</td><td>(Anne)</td><td>
</td><td>
</td></tr>
<tr><td>Transforming</td><td>(Violet)</td><td>
</td><td>
</td></tr>
<tr><td rowspan="2">Reforming</td><td>(Violet)</td><td>
</td><td>
</td></tr>
<tr><td>(Anne)</td><td>
</td><td>
</td></tr>
</table>

Table 3: Anne and Violet's Forms of Reorganizations of Relationships in Hot Air Balloon

Bridging. After having some time to freely explore the simulation themselves, Anne was asked if she noticed a relationship between the variables. She responded, "Yeah, when I was turning the flame up, it [the balloon] would go up every time I would turn it," showing that she reasoned covariationally about the size of the flame and the altitude of the balloon. The next day, Anne was asked about the relationship between the flame and the density of the air in the balloon. She explained, "Whenever you turn it down, the density becomes higher," reasoning covariationally about the relationship between the flame and the density of the air in the balloon. When she was later asked what the relationship between the altitude and the density was, she described the relationship, "When you turn the flame up, the altitude gets higher [turned the flame up in the simulation] and then the density gets lower. And then if you turn it down [turned the flame down], the altitude gets lower and the density gets higher." We interpret Anne's multivariational reasoning to be a result of bridging by merging the two covariational relationships to generate a more complex partial dependent multivariational relationship between the flame, altitude, and density of air in the balloon.

Transforming. Similar to Anne, when asked what she was able to control in the simulation, Violet first reasoned covariationally about the flame and the altitude of the balloon stating, "What I control is the flame to turn on [turned the flame up] to make the balloon go higher and the flame to turn down [turned the flame down] to make it lower." However, later when she was asked about the density of the air inside the balloon, Violet responded, "What happens is that when I go higher [turned the flame up], the density inside the balloon gets lower [used cursor to point to the density output], [turned the flame up more] but the temperature goes higher [pointed to the temperature output]." In contrast to Anne who bridged two covariational relationships, we interpret Violet's construction of an integrated multivariation relationship as a result of transforming her initial covariational relationship into a multivariation relationship by adding the quantities of temperature and density.

Reforming. As she continued to explore the simulation, Violet was asked about the relationship between temperature and density. Violet explained, "the hotter the air inside the balloon is, the more its density decreases." Here Violet reasoned about the covariational relationship between temperature and density. When asked to further explain, Violet said, "Because when you turn up the flame, it gets hotter, the density decreases, and it makes the balloon fly up higher." We interpret Violet's reasoning to show that she modified her previous integrated multivariation relationship using the new covariational relationship by reforming the relationship into a nested multivariational relationship.

Similarly, Anne also reorganized her reasoning to express different multivariational relationships after considering new covariational relationships. For example, she first reasoned that "for the temperature, when you turn it [the flame] down, it gets cooler. And then for the density, it decreases," constructing a multivariational relationship in which a change in one variable caused changes in two others. Later, after she had constructed the covariational relationship between temperature and altitude, we again prompted Anne to reason about all of the quantities. She responded, "When I turn up the temperature, the density starts getting low and then altitude, it shows how like the balloon is going up." Like Violet, we consider this excerpt to show that Anne had reformed her construction of the multivariational relationship into one in which changes in each of the quantities caused a change in the next in sequence.

CONCLUDING REMARKS

Our findings illustrate that students go through different mental processes, and thus different constructions and reorganizations as they construct meanings about multivariation. In this paper, we presented three different ways that students used to construct and reorganize meanings about the relationships of multiple quantities, namely bridging, transforming, and reforming (Tables 1-3). In all three simulations, we see examples of students engaging in a *bridging* form of reorganization in which they first constructed two covariational relationships and then merged these into a single multivariational relationship. However, in the Hot Air Balloon, we also see Violet engage in *transforming* her existing construction of a single covariational relationship into multivariation by reorganizing it to include the addition of new variables. In other words, she expanded her covariational relationship into a multivariational one. Also in the same simulation, both Violet and Anne engaged in *reforming* their initial multivariational reasoning after considering more of the covariational pairs that the larger nested relationship in the simulation was constructed from. This may indicate that the nature of nested relationships has some effect on students' progressions of multivariational reasoning.

We also noticed that in the two simulations, Bob's Life and the Hot Air Ballon, the four students constructed a multivariational relationship that matched the form of the underlying behavior of the simulation. However, in the "How high are those cumulus clouds???", Violet constructed the same relationship as the simulation while Anne only expressed the relationship with one of the three quantities held constant as a parameter. We conjecture that this may have been because of the different number of independent (i.e., controllable) variables in these cases. While each of the other simulations includes one independent variable, the "How high are those cumulus clouds???" simulation has two. This difference results in the simulations modeling multivariational relationships of different forms.

We thus believe that more research is needed on characterizing students' constructions and reorganizations in different types of multivariational situations. Finally, we consider that the nature of our questioning may have also influenced how students reasoned, since we drew their attention to successively increasing numbers of varying quantities. It may be possible that if we began by questioning students about multiple quantities, then they might engage in different forms of reasoning than we have described here.

Acknowledgements

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APPROACHES TAKEN BY PROSPECTIVE ELEMENTARY TEACHERS WHEN NOTICING CHILDREN'S ALGEBRAIC THINKING

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In this study, we adopt the perspective of Professional Noticing of Children's Mathematical Thinking to characterize the teacher knowledge of 21 prospective elementary teachers (PTs) when considering the algebraic thinking of their students. Specifically, our objective was to describe how PTs attend to, interpret, and decide about the strategies used by a 10-year-old child to solve the open equality 6 + 4 = ☐ + 5. The PTs participated in an early algebraic thinking content course and analyzed a video showing the solution strategies used by the child. The PTs had to reply to various prompts aimed at attending to the child's strategies, interpreting his understanding, and deciding how to modify the task based on the boy's understanding. We analyzed the PTs' responses through a content analysis following a data-driven approach. The main results show evidence that PTs followed two approaches when attending to and interpreting the child's strategies: a relational approach and an arithmetic one.

Various scholars have highlighted the importance of delving into how teachers perceive mathematical thinking and the learning processes of their students (e.g., Chapman, 2017). Consideration of mathematical thinking allows teachers to make informed decisions by applying their professional knowledge, which can be addressed through Professional Noticing of Children's Mathematical Thinking (Jacobs et al., 2010). The purpose of this specialized type of mathematics teacher noticing is to understand the characteristics of student thinking, interpret it and make evidence-based informed decisions. Specialized literature evidence that the expertise of noticing is learned and does not occur naturally during the teaching experience (e.g., Schack et al., 2013; Sherin et al., 2011). Therefore, it is important to address it during teacher training. This study focuses on algebraic thinking, a type of mathematical thinking. In spite of the growing body of research reporting on aspects of algebraic thinking in students aged 6-12, a more in-depth description is still needed on how PTs address learning when promoting algebraic thinking in children (Pincheira & Alsina, 2021).

This paper focuses on describing how PTs become involved in specific aspects of algebraic thinking in elementary school children (aged 6-12). Specifically, the objective of the study was to describe how PTs attend to, interpret, and decide about the strategies used by a 10-year-old boy to solve the open equality 6 + 4 = ☐ + 5.

THEORETICAL PERSPECTIVES

We focused on generalized arithmetic as an approach to algebraic thinking. From this perspective, the emphasis was on noticing regularities in arithmetic operations that can be generalized beyond specific numbers (Blanton et al., 2011). The equal sign takes on a key role, which can be addressed with a focus on a view: (a) *arithmetic*, involving the equal sign and immediately providing a result; or (b) *relational*, understood as a flexible approach to calculation where expressions are transformed using the fundamental properties (Carpenter et al., 2003; Knuth et al., 2006).

Professional Noticing of Children's Mathematical Thinking is considered a way of analyzing how teachers (in training or practicing) "see and make sense of classrooms in different ways and how particular types of experiences can support the development of their abilities to notice in particular ways" (Jacobs et al., 2010, p. 192). In particular, the focus was on analyzing the evidence of the student's mathematical thinking and making decisions at that moment. Concretely, this model considers a set of three interrelated skills: (a) attending to children's strategies, (b) interpreting children's understandings, and (c) deciding how to respond on the basis of children's understandings.

The study

The study presented is qualitative and exploratory. It follows Design-Based Research addressed through a Video Club (Sherin & van Es, 2009) in the context of an early algebraic thinking content course. The Video Club was composed of eight sessions. In the first session, we introduced the algebraic thinking model proposed by Blanton et al. (2011) and presented the Video Club as a work methodology. We have focused on sessions 2 and 3, aimed at generalized arithmetic. In both sessions, we played the same video showing David's strategies, a 10-year-old boy, to complete the open equality 6 + 4 = ☐ + 5. We chose this one of the boy's answers as it addresses the solution with two strategies: one focusing on calculation ("I put ten, but realized it was a mistake because I added 6 plus 4"), and another relational one ("we were checking that numbers gave the same result, 6 plus 4, and 5 plus 5").

Twenty-one Chilean PTs who were halfway through their training participated. The PTs watched the same video twice in succession at the start of the second and third sessions. Between the two sessions, which were a week apart, they addressed conceptual elements involved in generalized arithmetic as an approach to algebraic thinking, present in the Chilean curriculum, and how relational thinking is developed and used in the mathematics classroom. They all had the video transcription and were asked to react, in writing, to a different prompt (taken from Jacobs et al., 2010) in each session, as shown in Table 1.

Video Club Session	Noticing skills	Prompts
2	Attending	(1) Describe in detail what David did when answering the open equality.
3	Interpreting	(2) What can be said about David's understanding?
	Deciding	(3) How would you change the task to encourage David not to focus on calculation? What would you say or do?

Table 1: Prompts posed to PTs in each Video Club session

We conducted a content analysis following a data-driven approach of the PTs answers to the three prompts (see Table 1). This analysis promotes the development of categories based on data to be able to delve into how PTs attend to, interpret, and decide based on David's strategies.

RESULTS

We identified two approaches followed by PTs when attending to and interpreting David's strategies: *arithmetic* (Ar), focused on describing and/or interpreting David's calculations when filling in the blank (☐), or (b) *relational* (Rel), focused on describing and/or interpreting how David established equivalence relations between the equality elements and the equal sign. Regarding the deciding skill, we identified three different approaches when indicating how they would change the task: (a) focus on the equal sign (Eq), (b) focus on the operation properties (Op), and (c) focus on the calculation (Ar). Some students did not answer (NA). The main results are shown in Table 2.

	Noticing skills									
	Attending			Interpreting			Deciding			
Approach	Ar	Rel	NA	Ar	Rel	NA	Eq	Op	Ar	NA
Students	10	10	1	3	15	3	15	2	2	2

Table 2: Approaches evidenced in PT answers

In relation to *attending*, half of the PTs who described David's strategies followed an arithmetic approach, and the other half used a relational one. An example of the arithmetic approach is seen in the answer by PT_{02}: "He added 6+4. He realized he had made a mistake. He justified his mistake. He added 6+4 (…)". In this example, the PT described the calculations conducted. On the other hand, an example of the relational approach can be found in the answer by PT_{06}: "He wrote 10 since 6+4=10, but then he realized his mistake: a number which has to give the same result, 6+4=5+5. He was able to establish relations by understanding the meaning of the equal sign". In this example, we note how PT_{06} focused their description on the relationships between numbers ("establish relationships by understanding the meaning of the equal sign") while at the same time interpreting David's strategy. We found that the PTs who followed the relational approach tended to interpret spontaneously, unlike those following the procedural approach, which only narrated the calculation conducted.

Regarding the *interpreting*, most PTs followed a relational approach when interpreting and making sense of the strategy used by David. Another three PTs followed an arithmetic approach, and the remaining PTs did not answer. An example of an arithmetic approach when interpreting is that of PT_{10}: "He understands the operations, as in both cases he can calculate quickly and sufficiently. What he cannot develop so well is relational thinking, as he only sees the equal sign as part of a rigid operation". In this example, representative of this group of PTs, we found a more specialized language, including expressions such as "relational thinking" and "equal sign as part of a rigid operation".

With regard to the *deciding* based on David's understanding, 15 PTs showed an approach focusing on the role of the equal sign in the mathematical task. Thus, for example, A_{12} pointed out: "Initially, I would tell him to look at what there is on the other side of the = sign, because if he can see there is another mathematical expression it means he does not have to solve, rather it is an equality or equivalence. The fact there is a = does not necessarily mean solving with calculations, but rather finding the relationship". On the other hand, two PTs focused their decisions only on the properties of the operations, as PT_{17} did: "What I would do is help him focus on occupying the associative property". Finally, two PTs decided to jointly consider a focus on the equal sign and the properties, and two other PTs did not answer this prompt.

DISCUSSION AND CONCLUSIONS

With this study, we have identified how PTs attend, interpret, and decide about the strategies used by a 10-year-old boy to solve the open equality 6 + 4 = ☐ + 5. Specifically, we identified approaches followed by PTs when interacting with the child's reasoning. These findings allow highlighting three main contributions. First, several authors indicated it is necessary to characterize how teachers perceive the mathematical thinking of their students and to what extent they do it instead of focusing on the variety noticed (Jacobs et al., 2011). The approaches identified help obtain information on the PTs' view of the development of algebraic thinking in elementary education. The approaches go beyond identifying what PTs attend, interpret, or decide; they enable evidencing how these skills interact with the characteristics of algebraic thinking. The second contribution is jointly addressing the three noticing skills (Jacobs et al., 2010). Currently, the literature focuses primarily on attention and interpretation and rarely on decisions (König et al., 2022). Third, regarding the relational and the arithmetic approaches found in the PTs' answers when attending and interpreting, our results suggest that, while PTs had to describe in detail the children's strategies, it is difficult to separate description from interpretation, as highlighted by various authors (Jacobs et al., 2010). Furthermore, the approaches to observation and interpretation differ from those identified in the decision-making process, suggesting that the decisions made by PTs may warrant careful attention and consideration.

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UNDERSTANDING OF THE EQUAL SIGN: A CASE OF CHINESE GRADE 5

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Understanding the equal sign is a fundamental concept in early algebra. Recently, while there has been literature on Chinese students' understanding of the equal sign, the developmental progression of this understanding remains under-researched. This study further explored Chinese primary school students' understanding of the equal sign to contribute to addressing this gap. 237 grade 5 students were tested for their understanding of the equal sign. The results showed most students possessed a relational understanding, and a substantial number of them developed sophisticated structural thinking. This research also found that the exposure to simple equations was well mentioned by teachers as a key factor that supported students' conception of the equal sign.

BACKGROUND AND LITERATURE REVIEW

Students' misconception of the equal sign is widely reported in the literature: many students consider the equal sign as a symbol of displaying calculation results (i.e., an operational understanding) rather than indicating an equivalent relationship of both sides (i.e., a relational understanding) (Blanton et al., 2018). The relational understanding of the equal sign is a fundamental concept during students' progression from arithmetic to algebra. The narrow conception of the equal sign causes students difficulties in algebra, such as equation representation and solving (Blanton et al., 2018). Recently, a growing body of literature has been exploring Chinese students' understanding of the equal sign. For instance, Li et al. (2008) found that the majority of grade 6 students in China hold a robust relational understanding. Sun and Gu (2023) further investigated the pedagogical approach to introduce the equal sign in China. They identified a key contributor to students' relational understanding: the equal sign is first introduced to students in a quantity comparison context before they begin learning arithmetic operations in kindergarten. This approach provides students with a foundation of a relational view towards the equal sign before they begin primary school (Sun & Gu, 2023). However, the Chinese students' process of gaining a solid relational understanding is not without hiccups. As Sun et al. (2023) showed, by grade 3, only about half of the tested students held a robust relational understanding of the equal sign. They suggested that the extensive arithmetic drill in early primary grades contributed to students' conceptions of the equal sign reverted to the operational view. Therefore, there is still space to explore Chinese students' process of developing a relational understanding of the equal sign further. This study focuses on students who were at the start of grade 5. We first examined their relational understanding of the equal sign. We then tried to identify the possible factors that influenced students' conceptions. By doing this, this study contributes to a body of literature on Chinese students' learning continuum of development of relational understanding of the equal sign.

METHODOLOGY

237 grade 5 students, 110 boys and 127 girls, from S primary school in Changchun, Jilin Province, China, participated in this study. The context of the participating school and students is similar to those

in Sun et al. (2023) (e.g., similar SES and academic rankings). Students took a diagnostic test (elaborated below). Their responses will be coded against three categories of understanding of the equal sign suggested by Stephens et al. (2013). Six mathematics teachers were interviewed afterwards to seek their comments about the factors that contributed to or hindered students' relational understanding of the equal sign. The study was conducted at a time when students just started grade 5.

Instrument

Mathematics Equivalence Assessment Instrument [MEA] is a well-established tool for measuring students' understanding of the equal sign, and it can be used in cross-cultural contexts, including in China (Simsek et al., 2021). MEA comprises three types of problems: 1) structure evaluation, which requires students to determine whether a number sentence, such as 11=2+9, is true or false; 2) number sentence solving, which asks students to fill in the missing number in a number sentence, for example, 18+46 = __+47; 3) students' explanation of definition of the equal sign. Sun et al. (2023) adapted MEA to suit the context in China (e.g., change the numbers to better align with students' grade levels) and applied it to measure the first three grades students' understanding of the equal sign in China. This study used test items similar to those of Sun et al. (2023) since the contexts of the two research studies are similar. Students wrote the definition of the equal sign first. Then, students evaluated true/false for given number sentences. Finally, they were required to fill in the missing numbers in the number sentences. For number sentence evaluation and solving items, students were asked to write explanations about how they got answers. There were four types of number sentences for evaluation and solving, as shown in Table 1.

Number sentence type	Elaboration and example
"a + b = c"	Students evaluate true or false to number sentences such as 31+12=43 Students fill the missing number to the form __+35=91
"c = a + b"	Students evaluate true or false to number sentences such as 25=16+9 Students fill the missing number to the form 52=13+__
"a = a"	Students evaluate true or false to number sentences such as 41=41 Students fill the missing number to the form 23=__
"a + b = c + d"	Students evaluate true or false to number sentences such as 41+23=31+33 Students fill the missing number to the form 53+31=__+21

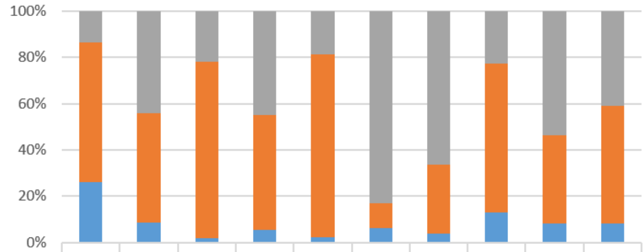
Table 1: Example Test Items for Number Sentence Evaluation and Solving

These four types of number sentences are canonical and non-canonical forms of number sentences which are widely used to test students' understanding of equal sign (Sun et al., 2023).

Coding process

Students' responses were coded based on three categories of understanding of the equal sign in Stephens et al. (2013). The first one is 'operational': students consider the equal sign as showing the results of the calculation carried out. The second one is 'relational-computational': students understand the equal sign as indicating equivalence of both sides, but they still need full calculations to demonstrate this equivalence. The third one is 'relational-structural', which means that students can apply relationships among quantities to show the equivalence with minimum calculations. Both the second and third types can provide evidence that students possess relational understanding. In this study, for the definition of the equal sign, a response was coded as ‘operational’ if a student stated that the equal sign means “adding numbers”, “answers”, or “totals”; if a student expressed that equal sign meant equivalence of both side by using specific calculation examples, it was coded as ‘relational-computational’; if a student explained equal sign meant equivalent quantities of both sides in general, it was coded as ‘relational-structural’. For number sentence solving, for example, when solving “7+3=__+4”, if a student filled in 10 or 14 for the missing number, it was coded as ‘operational’; if students calculated 7+3=10 and then 10–4=6 for the missing number, it was coded as relational-computational; if a student recognised that 4 is 1 more than 3 and so the missing number should be 1 less than 7 so it is 6, it was coded as ‘relational-structural’. The coding procedure for number sentence evaluation items was similar.

RESULTS AND DISCUSSION



	Definition of equal sign	"a+b=c" True/False	"a+b=c" Solving	"c=a+b" True/False	"c=a+b" Solving	"a=a" True/False	"a=a" Solving	"a+b=c+d" True/False	"a+b=c+d" Solving	average
relational-structural	13.5	44.3	21.9	45.1	18.6	83.1	66.2	22.8	53.8	41.0
relational-computational	60.3	47.3	76.4	49.4	79.3	10.5	30.0	64.4	38.2	50.6
operational	26.2	8.4	1.7	5.5	2.1	6.3	3.8	12.9	8.0	8.3

operational relational-computational relational-structural

Figure 1: Distribution of Students’ Responses to Test Items against Categories (%)

Figure 1 shows the majority of students achieved either level of relational-computational or relational-structural. Above 74% and 87% of responses fall into these two categories in the definition item and across all other test items, respectively. This result tends to indicate that by the start of grade 5, most students have developed a relational understanding of the equal sign. Furthermore, a significant amount of students have demonstrated a structural view towards the number sentence (e.g., 53.8% of responses used relational-structural thinking to solve the number sentence type “a + b = c + d”), which can be considered as an emergence of the early algebraic thinking (Stephens et al., 2013). Sun et al. (2023) documented that only nearly half of grade 3 participating students possessed an operational understanding of the equal sign. While the participant cohort in this study was different from Sun et al. (2023), considering that the research contexts (e.g., schools, students) of the two studies were similar, it could be argued that compared to grade 3 students in Sun et al. (2023), there is an enhancement in students’ conception of equal sign by the start of grade 5. For instance, regarding the definition of the equal sign item, in this study, 74% of grade 5 students provided the ‘relational’ explanation, compared to only 40.4% of grade 3 students in Sun et al. (2023). Six teachers were interviewed to suggest what they considered to promote students’ relational understanding of the equal sign. Almost all teachers mentioned the exposure to simple equations (e.g., x+1=2, 2=x–1) in grade 4 contributed to reinforcing students' relational conception of the equal sign. Some teacher interview excerpts are shown below,

> Teacher A: When introducing equations, we used the balance model. Students visualised the similarity between the abstract equations and the concrete balance. So, they easily comprehend that the equal sign indicates an equivalent relationship of both sides, like the balance beam.
>
> Teacher B: These students have learnt the concept of equivalence and equal sign in kindergarten by comparing quantities. When they were learning simple equations and saw the balance model, the prior knowledge about the equal sign was reactivated and reinforced.

Teachers considered that the resemblance between the balance (beam) and equation visually supports a conception that the equal sign refers to the equivalence of both sides. As mentioned by Teacher B, the exposure to simple equations with the balance model can visually 'reactivate' and 'reinforce' that students perceive the equal sign as an indicator of equivalence of both sides, given that students had built a foundation for relational understanding in the early years. This process was further elaborated by teachers, stating that when learning equations, students participated in a hands-on play with the balance, seeing the beam tilting if one side was heavier and observing it balancing again when they adjusted the weights to make both side's weight equal. This dynamic process facilitated them to attend to the concurrent changes in both sides of the balance, hence pressing the conception of the equal sign that represents the equivalence of both sides. Teacher B further commented, "Students had learnt the equal sign in the context of comparing quantities at earlier ages, but this comparison was static. In contrast, the dynamic play of the balance model could reinforce the meaning of the equal sign more". Furthermore, three teachers mentioned that when learning equations, compared to earlier grades, students were provided with more opportunities to experience non-conventional forms of arithmetic operations (e.g., 5=x–2), helping them depart from an operational mindset. Traditionally, students need to have a relational understanding of the equal sign first before they can understand the equation. In the case of this study, it appears that since students had been laid with a foundation of the relational conception in earlier years, the introduction to simple equations, with the assistance of the balance model, can help them consolidate the relational understanding of the equal sign in turn.

CONCLUSION

The findings of this study tend to suggest that by the start of grade 5, Chinese primary students have developed a robust relational understanding of the equal sign. One contributing factor could be students' exposure to simple equations with the balance model in grade 4, which could strengthen their relational view towards the equal sign that had already emerged in kindergarten. This study is explorative, so further investigation is worth conducting to understand more details about how simple equations enhance students' conception of the equal sign, possibly with some in-depth case studies.

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DEVELOPING RELATIONAL THINKING WITH SUBTRACTION AS DIFFERENCE: THE CASE OF TIMOTHY

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A foundational goal of early algebraic thinking is the development of a relational view of equivalence. In this study we explored Year 3 & 4 (9-11-year-old) students' responses in individual interviews to subtraction tasks designed to elicit their attention to the compensation property of equality. The context was comparing pairs of vertical towers of blocks. Video data were analyzed using the Student Noticing Framework for evidence of students' relational thinking, using concrete, contextual, and symbolic representations of subtraction as difference. In this paper we share our case study of Timothy. We provide evidence that physical towers of blocks elicited Timothy's attention to, and expression of, the subtraction-compensation property but highlight his initial difficulties in attending to the direction of comparison between matching minuends and subtrahends. His subtraction fluency with smaller numbers seemed to support him in experimenting with relational thinking, through being able to double-check his ideas with calculations.

Giving children the opportunity to learn generalized arithmetic through seeing the structure and properties of equations is important early on in their mathematics learning (Kaput, 2008). Kaput (2008) argued that children need to learn to abstract structures and systems from computations and relations at the time they are learning arithmetic: to look for and describe patterns and to use properties to develop and justify a variety of strategies. Such an algebraic lens may support the development of computation fluency during primary school (Chesney et al., 2018), while also building the necessary algebraic thinking for future algebra learning in secondary school (Kindrat & Osana, 2018). Previous research has highlighted some of the difficulties that children experience in learning to think relationally, particularly using the subtraction-compensation property of equality (e.g., Cooper & Warren, 2011; Shumway, 2018). This property can be described as follows: what you add to (or subtract from) one operand must be added to (or subtracted from) the other operand for the difference to remain the same [if $a – b = d$, then $(a + c) – (b + c) = d$] (Russell et al., 2011). To date, representations used to help children attend to, and explain, the compensation properties of equality have found to be lacking.

In this study we sought insights into middle-primary students' attention to and expression of generality with the subtraction-compensation property of equality through using a hands-on tool: vertical towers of blocks. We explored the potential of towers of blocks to support children's relational thinking with tasks based on a 'comparison' (rather than 'take-away') model of subtraction (Usiskin, 2007). With the comparison model, the quantity $a – b$, the 'difference', tells how much b is less than the quantity a. With the take-away model, $a – b$, the 'remainder', is the quantity left when quantity b is taken away from an original quantity a. We hypothesized that students' documented difficulties with the subtraction-compensation property may be due, at least in part, to their limited experience of the comparison model of subtraction (i.e., viewing subtraction as difference) and integrating this knowledge with their understanding of the take-away model of subtraction. The take-away model of subtraction has traditionally received considerably more attention in school mathematics than the comparison model (Selter et al., 2012).

BACKGROUND

Despite notable differing conceptualizations of early algebra, there is widespread consensus that early algebraic thinking involves (i) deliberate generalizing and expressing generality, and (ii) reasoning based on generalizations (often as a separate endeavour). Algebraic thinking does not necessarily require communicating generalizations using conventional alphanumeric symbolization. It can be also communicated semiotically, such as through speech, gestures, and written markings (Lins & Kaput, 2004; Kieran, 2022; Radford, 2011).

Kieran (2022) categorized and synthesized research activity around children's early algebraic thinking related to generalized arithmetic and referred to the general dimension of 'relational thinking'. Relational thinking involves "seeing and expressing structure and properties within numbers, operations, and expressions" (Kieran, 2022, p. 15). Consistent with this definition but more specific, we refer to relational thinking as thinking pertaining to properties of equality (Wilkie & Hopkins, 2024). With subtraction, students' relational thinking can be evidenced in their attention to and expression of the compensation property of equality—how increasing or decreasing both minuend and subtrahend in a subtraction expression by the same amount results in an equivalent expression (Russell et al., 2011). Relational thinking about equality is distinguished from operational thinking about equality, which is primarily computational in nature (e.g., 15 – 3 = 16 – 4 because both sides of the equation equal 12).

Tools for developing relational thinking

Children visualizing and learning to coordinate both spatial and numeric structures play an important role in early algebraic thinking (Radford, 2011). In generalized arithmetic, "it is the meaning of the operations, as represented visually in diagrams, manipulatives and so on, that forms the basis of justification of claims of generality" (Schifter et al., 2008, p. 443).

In early algebra research, physical materials, and visual representations, such as pan balances have been used to helping children develop a relational interpretation of the equal sign (e.g., Stephens et al., 2021). Australian researchers investigated the use of unnumbered paper strips and number lines for middle primary students to learn the compensation properties of equality but found that students became confused when moving from addition (termed 'do the opposite' strategy) to subtraction ('do the same' strategy). They surmised that the different strategy for the subtraction-compensation property as compared to the addition-compensation property (i.e., do the same versus do the opposite) was problematic for developing this type of relational thinking (Cooper & Warren, 2011). We think that a likely contributing factor to the students' difficulty was that the comparison model of subtraction was not clearly represented for them.

In this current study we investigated the affordances of, and difficulties with, the use of vertical towers of blocks for eliciting relational thinking with subtraction. We chose this tool since it provides opportunities for children to make sense of subtraction modelled as difference with a familiar real-life representation that combines continuous attributes (i.e., height) and discrete quantities (number of blocks), in line with Lins and Kaput (2004) [citing Confrey (1991)], who suggested children work with discrete and continuous quantities in complementary ways.

In school mathematics, teachers typically emphasize the take-away model of subtraction more than the comparison model (Selter et al., 2012; Usiskin, 2007) even though both are important for understanding the complementary properties of additive structures and underpin the variety of flexible strategies for subtraction, including indirect addition (Selter et al., 2012). In this study we developed a sequence of 15 tasks based on the comparison model of subtraction, involving concrete, numeric and symbolic representations, and increasing in difficulty according to the size of the minuend, subtrahend, and/or difference (single digit to three-digit numbers). This paper shares the case study of a Year 4 student, Timothy (pseudonym), and addresses the following research question: *How do students evidence relational thinking in the context of modelling subtraction as comparison with pairs of vertical towers of blocks?*

RESEARCH DESIGN

In a qualitative collective case study, 22 9-11-year-old students' interview responses to a sequence of subtraction tasks were analyzed in depth using the Student Noticing Framework (Lobato et al., 2013). This framework has been used in prior research on students' algebraic thinking (e.g., Wilkie, 2022). The unit of case study analysis (Creswell, 2013) was each student, who were selected randomly to include a wide range of prior levels of understanding about subtraction. Semiotic data (verbalizations, constructions, written markings, and hand gestures) were collected from video recordings and written work samples.

The 15 tasks were designed using theoretical perspectives and empirical findings from the literature on embodied visualization for algebraic thinking (e.g., Radford, 2011) and relational and structural thinking (e.g., Mason et al., 2009). Students were involved in physically constructing and drawing towers of blocks, comparing numeric pairs of towers with specified numbers of blocks (e.g., '10 blocks & 8 blocks'), and then assessing true/false equivalence questions with subtraction expressions and completing matching open questions for finding equivalent expressions. Carpenter and colleagues (2003) recommended true/false and open question formats for developing early algebraic thinking with generalized arithmetic.

Initial tasks making and drawing towers were included to help students visualize subtraction as the difference between tower heights. We assumed that children of this age may not have developed this meaning for subtraction, given the predominance of the take-away model, but would likely have had some prior experience of comparing lengths and numbers of objects. Later tasks involving written numeric pairs of towers were intended to emphasize the comparison model of subtraction in a contextualized yet written representation. The remaining tasks involved conventional subtraction expressions and equations (e.g., 'True or false': 34 – 28 = 30 – 24 and 'Fill in the numbers': 34 – 28 = __ – __) but were still worded as the difference in tower heights. These tasks increased in the size of minuend and subtrahend, with the intent of providing repeated opportunities for students to express generality by using the subtraction-compensation property of equality.

THE CASE OF TIMOTHY: RELATIONAL THINKING WITH SUBTRACTION

In the following three sub-sections, findings on how Timothy evidenced relational thinking with subtraction tasks are shared.

Adjusting physical towers of blocks

Timothy was first asked to make two towers of blocks with different heights (using joinable Unifix blocks). He made towers of two blocks and one block, standing them vertically in front of him. When asked about their difference in heights, he explained (and also wrote a sentence) that Tower 1 was one block taller than Tower 2. Timothy was then asked to make two towers with the same difference in heights, and he made towers of three blocks and two blocks, placing them in front of his original towers (corresponding with taller tower on the left). He gestured with his hands and explained:

> Tower 1 has been added one block, that would make a tower of three [gestured from original to new tower]. So I added one block, just like Tower 2 has, that's been added one block [gestured again]. (Q2)

Timothy, compared to some other student participants, built towers with minimal numbers of blocks and explained, "I am just going to start a bit easier". He evidenced adjusting his original towers rather than calculating the difference. His response is suggestive of both attending to the subtraction-compensation property through his gesturing and expressing it through explaining: "I added one block, just like Tower 2 has." His very short towers hinted at his penchant for finding a quick way to answer, which was noticeable in his later task responses as well.

Moving from 'minusing' to noticing movements

In the next three interview tasks (Q3 - Q5), Timothy was asked to identify the pair of towers that did not belong in a list of four pairs. The tower heights were represented numerically as amounts of blocks (e.g., '10 blocks & 8 blocks', '11 blocks & 9 blocks' etc.) in the first two tasks, with each pair written underneath each other vertically. He circled the correct answers almost immediately, using both the language of difference and of take away ("minusing"):

> All of these three have a difference of two, except this one [points to '7 blocks & 4 blocks'], which is a difference of three. (Q3)
>
> These three are the same. This one is minusing 11 [points to '30 blocks & 19 blocks'], minus 10 [points to 25 & 15], minus 10 [points to 20 & 10], minus 11 [points again to 30 & 19], minus 10 [points to 15 & 5] (Q4)

His prompt and accurate responses indicated fluency with subtraction involving smaller numbers and being comfortable with both models of subtraction (comparison and take away) when working with numeric representations of pairs of towers.

In Q5, the task representation changed to symbolic subtraction expressions. The interviewer (first author) explained that the task was still the same (to find the pair of towers that doesn't belong) but the tower heights were written with the subtraction symbol, saying, "I've written the towers down this way with the subtraction symbol but it can still mean the difference between these two numbers." Timothy used and explained an indirect addition strategy to solve 35 – 8 and then also described it in terms of "minusing":

> With this one [35 – 8] I'd go 8, 18, 28 and then 29, 30, … 35 [counts by ones on fingers], so adding 7 and 20, 27. So this would be minusing 27 to get 8. (Q5)

His use of language for both indirect addition and take away are suggestive of conceptual links between subtraction as difference and subtraction as take away, even with the shift to symbolic expressions. This was unlike some other participants, who initially used indirect addition with numeric representations, but then reverted to take away when presented with symbolic expressions. Timothy repeated the same indirect addition strategy for 34 – 9. He then recognized immediately that 30 – 3 was 27, explained he wouldn't need to work out the last one, and then correctly circled 34 – 9 as not belonging. As with the very short towers he made at the beginning of the interview, Timothy demonstrated an interest in answering a task with speed. This was supported by his computational fluency with smaller numbers.

The remaining tasks in the interview (Q6 - Q15) involved pairs of tasks about the same subtraction expression: four true/false assessments and three open-ended equations. For the first pair of tasks relating to 20 – 16, Timothy evidenced consistent use of operational thinking by taking away four to both assess different expressions and to create equivalent expressions. For example, he said when assessing 20 – 16 = 22 – 14:

> This would not be [true], because you'd need to minus four from 20 to get 16 but if you minus four from 22, you only get 18, not 14. So, it has to be false. (Q6)

For creating equivalent expressions for 20 – 16, Timothy explained that he chose a number and then "minused four" (and he wrote 50 – 46 and also 47 – 43).

With Q8 (see Figure 1) Timothy explained, "This time I'll do something different because the numbers are quite close." Rather than calculating and comparing the differences (evidencing operational thinking), as he had done before, he focused instead on the changes between minuends and subtrahends. This change in focus is suggestive of his attempting relational thinking. However, Timothy did not initially attend to the direction of the changes, only the magnitude. He reasoned,

> Since this has been added one [gestured from 33 to 34], and this has been added one [gestured from 28 to 29], they should be the same. (Q8 Part 1)

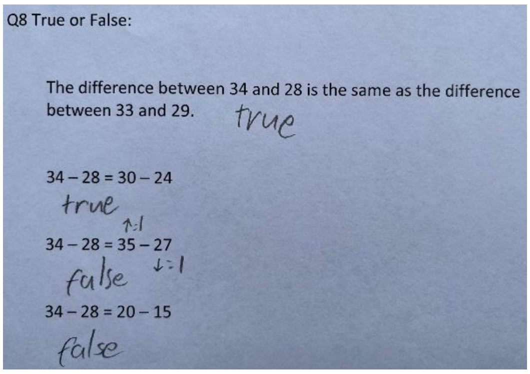

Figure 1: Timothy's shift to noticing magnitude and direction of changes

It is possible that because Part 1 of this question was worded as a statement (to re-emphasize the comparison model), the pairing of minuends and subtrahends may have been obscured for Timothy and he only noticed the change of 'one'. Unlike some other participants, Timothy did not at this stage double-check his answer with operational thinking and pick up his error.

In Q8 Part 2, when the representation became a symbolic equation (34 – 28 = 30 – 24), Timothy reverted to calculating the differences on each side of the equation. He again calculated the differences with Part 3 (34 – 28 = 35 – 27) before the interviewer asked him if he could use the same type of thinking he used with Part 1 (the written statement). He replied, “Yes, you could” and then reflected:

> Since this has been moved up one [writes upward arrow with ‘:1’ above 35] but this has been moved down 1 [writes downward arrow with ‘:1’ underneath 27] that would add on 2. (Q8 Part 3)

His written markings, including arrows, evidenced relational thinking with attention to both magnitude and direction of the changes, with correct pairing of minuends and subtrahends. Yet for the remainder of Q8 and Q9, Timothy reverted to calculating the difference, suggestive of operational thinking. It is possible that with the size of the numbers involved in the tasks and his evident fluency with subtraction, it would be more efficient for him to do so. As with Timothy, some other participants were found to attend to magnitude only in their initial attempts at relational thinking, but unlike Timothy, they did so both with written sentences and symbolic equations. They shifted to focusing on magnitude and direction, usually after double-checking their answers with calculations (operational thinking) and self-correcting. Another student evidenced relational thinking with physical towers of blocks but did not attend to magnitude with the written representations.

Experimenting with relational thinking and negative numbers

It was with Q10, when the subtrahend and minuend were (deliberately) two-digit numbers with a large difference, that Timothy chose to try relational thinking again. He explained (of the difference between 92 and 38 versus 93 and 40):

> I’d straight away go false because you’ve added one here but you’ve added two here. I wouldn’t know the difference, but I’d know it’s false. I only need to answer the question. (Q10 part 1)

Interestingly, unlike Q8 Part 1, Timothy now paid attention to both magnitude and direction with this written statement (see Figure 2).

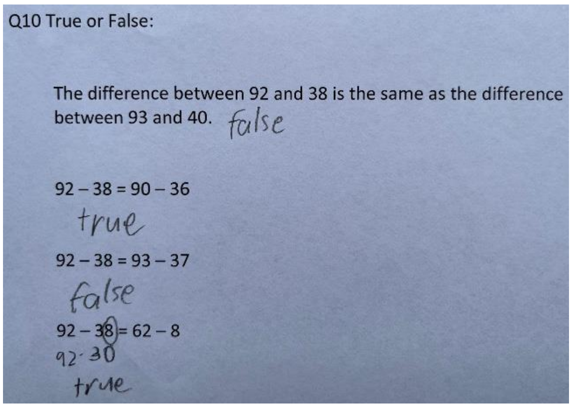

Figure 2: Timothy’s attention to direction and magnitude

It is not clear why Timothy did not have trouble using relational thinking with this written statement, as he had earlier with Q8. It is possible that having worked with numerous symbolic equation tasks in-between, he now interpreted it as intended—as an equation in written format. Unlike Q9, he evidenced choosing to attempt relational thinking again, possibly because it would be quicker to do so than use operational thinking to calculate both sides of the equations.

Surprisingly, with Q11, Timothy did not continue using relational thinking, even though it was the same pair of numbers as Q10 (92 – 38). Instead, to create his own equivalent expressions, he attempted indirect addition to calculate the difference. This time he counted by tens from 38 to 88 but subtracted 4 from 88 rather than added 4 to 50 and ended up with 84 (incorrect; not 54). Consequently, he wrote 84 – 0 for his first equivalent expression and asked if that was ok. The interviewer (rather than correcting the mistake yet noticing his use of zero) responded that he could even use negative numbers if he wanted to, like towers with underground levels. He reflected:

> I think about it as like trees and roots. I'm just going to go down [writes 83 – -1 and then 82 – -2]. Since I am minusing 1 every time [gestures to minuends], I have to minus one every time [gestures towards subtrahends]. (Q11)

It was lovely to hear Timothy adapt the towers analogy with underground levels for negative numbers to his own analogy of trees with roots. Although his initial subtraction calculation was inaccurate for this question, Timothy evidenced attempting relational thinking with expressions involving negative numbers, which was intriguing. It was also suggestive of his making conceptual links to some prior knowledge. The remaining tasks in the interview involved 3-digit numbers and Timothy continued to evidence relational thinking in both assessing and in creating equivalent expressions efficiently.

CONCLUSION

Timothy's case provided evidence of the potential of vertical towers as a tool for developing relational thinking with subtraction modelled as comparison. He attended to and expressed the subtraction-compensation property of equality when adjusting physical towers of blocks to keep their difference in height constant. In the tasks involving written statements, Timothy initially evidenced only attending to the magnitude of changes to matching minuends and subtrahends, and not direction. Yet this was only temporary and may have been an inadvertent task design issue since writing equations in sentences may have obscured the direction of changes. His fluency with smaller numbers and flexible use of subtraction strategies appeared to support his connecting of both models of subtraction (comparison and take away) and his relational thinking. His penchant for finding a quick strategy led to his choosing operational thinking for tasks where it seemed more efficient than attempting relational thinking. The task design, in not requiring calculation of the actual difference and increasing in the sizes of the numbers involved, may have played a role in persuading him to attempt relational thinking. Timothy demonstrated searching for an efficient way to solve different problems and in doing so he demonstrated flexibility.

There is more to understand about the process of students attending to and expressing generality when learning generalized arithmetic. This study contributes qualitative evidence that vertical towers of blocks are a useful representation for supporting children's attention to the subtraction-compensation property, and the need for further research on task design and sequencing to encourage students to develop relational thinking.

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