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Bank insolvency risk, Z-score measures and unimodal returns: A refinement

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ABSTRACT

We develop refined probability bounds for bank insolvency risk measures based on the Z-score, analogous to those given by Cantelli's inequality under the additional assumption of unimodality of returns, drawing on the one-sided Vysochanskii-Petunin inequality. Illustrating empirically for US banks, we argue that (i) unimodality of returns is not an overly restrictive assumption in this context, and (ii) the refined measures provide a less conservative alternative to insolvency probability bounds drawing on the (two-sided) Vysochanskii-Petunin inequality, particularly for banks with higher levels of insolvency risk.

1. Introduction

The Z-score is frequently used to assess a bank's probability of insolvency;[1] the metric is generally chosen because it is simple, intuitive, and requires only easily available accounting data.[2] While it is commonly attributed to Boyd and Graham (1986), Hannan and Hanweck (1988) and Boyd et al. (1993), a more refined probabilistic foundation for it was given in Lepetit and Strobel (2015), drawing on the Cantelli (1928) inequality, the one-sided equivalent of the Chebyshev (1867) inequality.

For the unimodal returns case, Strobel (2011) earlier presented upper bounds of the probability of insolvency drawing on the (two-sided) Vysochanskii and Petunin (1980) inequality; in this paper, we develop refined versions of such bounds using the one-sided Vysochanskii-Petunin inequality derived in Mercadier and Strobel (2021). Illustrating empirically for a large sample of US banks, for the period 2001q1-2020q1, we then firstly demonstrate that unimodality of returns is not an overly restrictive assumption in this context. As a consequence, we then argue that the refined measures provide an appropriate, less conservative alternative to insolvency probability bounds drawing on the (two-sided) Vysochanskii-Petunin inequality, a refinement that proves particularly significant for banks with higher levels of insolvency risk. Our results might, therefore, prove useful for a large range of practical applications in the area of banking and financial stability where more precise measures of a bank's probability of insolvency are relevant.

Section 2 now derives our refined, Z-score based insolvency probability bounds for unimodal returns; Section 3 provides an empirical illustration of our results; and Section 4 concludes the paper.

2. Refined insolvency probability bounds with unimodal returns

With car a bank's capital-asset ratio and roa its return on assets, the bank is considered insolvent if $(car + roa) \leq 0$; we want to assess the corresponding probability of default $\mathbb{P}(roa \leq -car)$. Generally, if roa is a random variable with mean μ_{roa} and variance σ^2_{roa}, and defining the Z-score as $Z \equiv (car + \mu_{roa}) / \sigma_{roa}$, this implies

$$\mathbb{P}(roa \leq -car) = \mathbb{P}\left(\frac{roa - \mu_{roa}}{\sigma_{roa}} \leq -Z\right).$$

For roa a unimodal random variable, Strobel (2011) draws on the (two-sided) Vysochanskii-Petunin inequality to obtain upper bounds of this probability of insolvency as

$$\mathbb{P}(roa \leq -car) \leq \begin{cases} \frac{4}{9}Z^{-2} & \text{for } Z \geq \sqrt{\frac{8}{3}}, \\ \frac{4}{3}Z^{-2} - \frac{1}{3} & \text{otherwise.} \end{cases} \quad (1)$$

However, drawing on the one-sided Vysochanskii-Petunin inequality, derived in Mercadier and Strobel (2021) for unimodal random

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[1] For some recent papers using this methodology, see e.g. Dang and Nguyen (2022), Klein and Weill (2022), Mateev et al. (2022), Bayeh et al. (2021), Nyola et al. (2021), Ardekani et al. (2020), Berger et al. (2017), Mare et al. (2017), Berger et al. (2016), Hakenes et al. (2015), Delis et al. (2014), Fang et al. (2014), Fu et al. (2014), Michalak and Uhde (2012).

[2] Note this measure is distinct from the Altman (1968) Z-score used in the corporate finance literature.

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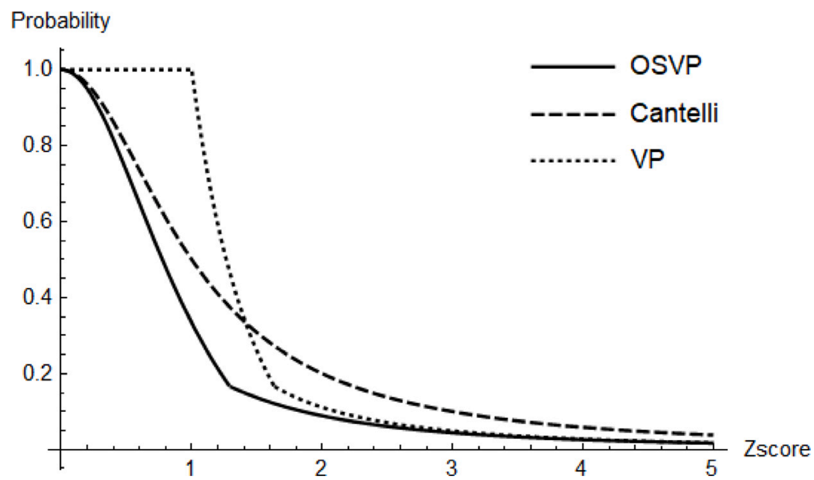

Fig. 1. (Theoretical) insolvency probability bounds for Cantelli's vs. one & two-sided Vysochanskii-Petunin inequalities (*OSVP & VP*), for range of Z-scores.

variable X as

$$\mathbb{P}(X - E[X] \geq r) \leq \begin{cases} \frac{4}{9}\frac{V(X)}{r^2 + V(X)} & \text{for } r^2 \geq \frac{5}{3}V(X), \\ \frac{4}{3}\frac{V(X)}{r^2 + V(X)} - \frac{1}{3} & \text{otherwise.} \end{cases},$$

we can obtain refined upper bounds of this probability of insolvency as

$$\mathbb{P}(roa \leq -car) \leq \begin{cases} \frac{4}{9}\left(1 + Z^2\right)^{-1} & \text{for } Z \geq \sqrt{\frac{5}{3}}, \\ \frac{4}{3}\left(1 + Z^2\right)^{-1} - \frac{1}{3} & \text{otherwise.} \end{cases} \tag{2}$$

Fig. 1 graphs the relationship between Z-scores and the insolvency probability bounds given by Cantelli's vs. the one and two-sided Vysochanskii-Petunin inequalities.[3] We note the significantly tighter bounds provided using the one-sided Vysochanskii-Petunin inequality.

3. Empirical illustration

We now illustrate how the insolvency probability bounds defined by Eqs. (1) and (2) (as well as in Footnote 3) compare when taken to the data. For this, we examine a dataset based on quarterly call reports for US banks, extracted from Wharton Research Data Services. We compute time-varying Z-scores using moving mean and standard deviation estimates for *roa* (with window width twelve quarters),[4] combined with current period values of *car*. We retain all banks for which we are able to construct a contiguous set of time-varying Z-scores of at least 20 quarters for the period 2001q1-2020q1; this results in data for 7493 banks in total.

We first test for unimodality in the returns series, for each of the 7493 banks and their respective full available samples in the period 2001q1-2020q1 (representing, on average, 37.6 quarters). We apply the widely used Hartigans' dip test for unimodality (Hartigan & Hartigan, 1985),[5] which is robust to outliers; resulting p-values are compared to the usual threshold of $\alpha = 0.05$. Our results show that the null hypothesis of unimodality is not rejected in 99.03% of all cases;[6] we would argue that unimodality of returns might not represent an overly restrictive assumption for many practical applications in this context.

Table 1 then gives descriptive statistics for Z-scores and the respective insolvency probability bounds considered. We note that the average relative difference between the insolvency probability bounds given by the one & two-sided Vysochanskii-Petunin inequalities is 1.567%, with a maximum value of 199.912%. For banks with the lowest 10% of Z-scores, i.e. banks with a more pronounced risk of becoming insolvent, the average relative difference between the two measures rises substantially to 14.184%. The average relative difference between the insolvency probability bounds given by Cantelli's and the one-sided Vysochanskii-Petunin inequality is 124.409% overall, with a maximum value of 125%;[7] this illustrates the significant refinement in upper bounds of the probability of insolvency achievable when unimodality of returns can be assumed.[8]

Our empirical illustration for US banks thus demonstrates that (i) unimodality of returns is arguably not an overly restrictive assumption in this context, and (ii) the refined probability bound drawing on the one-sided Vysochanskii-Petunin inequality developed in this paper provides a less conservative alternative to probability bounds drawing on the (two-sided) Vysochanskii-Petunin inequality, with the improvement being most significant for banks with higher levels of insolvency risk.

4. Conclusion

We develop refined probability bounds for bank insolvency risk measures based on the Z-score, analogous to those given by Cantelli's inequality under the additional assumption of unimodality of returns, drawing on the one-sided Vysochanskii-Petunin inequality. Illustrating empirically for a large sample of US banks, we then argue that unimodality of returns is not an overly restrictive assumption in this context, and that, hence, the refined measures represent an appropriate, less conservative alternative to insolvency probability bounds drawing on the (two-sided) Vysochanskii-Petunin inequality, of particular relevance for banks with higher levels of insolvency risk. As a consequence, they might prove useful for a wide range of practical applications in the area of banking and financial stability where greater precision in the measurement of a bank's probability of insolvency is important.

[3] The corresponding probability bound using Cantelli's inequality is $\mathbb{P}(roa \leq -car) \leq \left(1 + Z^2\right)^{-1}$, see Lepetit and Strobel (2015).

[4] This follows Boyd et al. (2006) and DeYoung and Torna (2013); it corresponds to the commonly used annual frequence of three years (see Lepetit and Strobel 2013).

[5] Note that it is commonly used for cluster analysis (see e.g. Adolfsson et al. 2019, Banerjee et al. 2017, Garcia-Dias et al. 2020); more recent unimodality tests focus on testing unimodality in a multivariate set-up (see e.g. Ahmed and Walther 2012, Burman and Polonik 2009, Siffer et al. 2018).

[6] For a threshold of $\alpha = 0.01$, this rises to 99.81% of all cases.

[7] For the banks with the lowest 10% of Z-scores, the average relative difference between the two measures is slightly lower at 119.092%; see also Fig. 1.

[8] Note that while normality of returns, as assumed by e.g. Boyd and Graham (1986) and Strobel (2010) in this context, would in theory provide even tighter probability bounds in the form of $\mathbb{P}(roa \leq -car) = \Phi(-Z)$, it is generally considered a poor description of actual return series in practice.

Table 1
Comparison of insolvency probability bounds given by Cantelli's and one & two-sided Vysochanskii-Petunin inequalities, for sample of US banks (N=7493) & period 2001q1-2020q1.

All bank observations (n=281801)

	mean	sd	min	max
Zscore	41.239	33.384	0.000	670.598
Prob(Cantelli)	0.011	0.063	0.000	1.000
Prob(VP)	0.010	0.077	0.000	1.000
Prob(OSVP)	0.007	0.053	0.000	1.000
Rel. diff. Cantelli/OSVP (%)	124.409	7.822	0.000	125.000
Rel. diff. VP/OSVP (%)	1.567	11.689	0.000	199.912

Bank observations with lowest 10% Z-scores (n=28181)

	mean	sd	min	max
Zscore	5.357	2.527	0.000	9.283
Prob(Cantelli)	0.099	0.177	0.011	1.000
Prob(VP)	0.090	0.227	0.005	1.000
Prob(OSVP)	0.062	0.158	0.005	1.000
Rel. diff. Cantelli/OSVP (%)	119.092	24.092	0.000	125.000
Rel. diff. VP/OSVP (%)	14.184	34.483	0.000	199.912

Note: Z-scores computed using moving mean & standard deviation estimates for return on assets (window width twelve quarters), combined with current period values of capital-asset ratio; OSVP & VP are short for one & two-sided Vysochanskii-Petunin inequalities; relative difference between Cantelli & OSVP bounds is defined as 100*(Prob(Cantelli)/Prob(OSVP)-1), and similarly for VP & OSVP bounds.

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