A causal and stable reduced-order model for linear high-frequency systems

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***Abstract*— With the ever growing complexity of high-frequency systems in the electronic industry, formation of reduced-order models of these systems is paramount. In this contribution, two different techniques are combined to generate a stable and causal representation of the system. In particular, Balanced Truncation is combined with a Fourier Series Expansion Approach. The efficacy of the proposed combined method will be shown for several examples.**

***Index Terms*— High speed systems, Model Reduction, Fourier Series Expansion, Balanced Truncation**

I. INTRODUCTION

In many branches of engineering, there is an ongoing need for efficient and effective model-order reduction techniques to counter the ever-increasing complexity of simulations. From control system design to RF Integrated Circuit design and optimisation, the formation of reduced order models has, for a long time, been a focus of attention for research [1 and references therein].

One popular category of reduction methods for linear systems are Krylov subspace methods e.g. [2]. These methods can handle large systems and are numerically efficient. However, there is no error bound for them and they generate non-optimal models. On the other-hand, Balanced Truncation and optimal Hankel model reduction [3] have global error bounds, but their associated computational requirements for their traditional implementations render them unsuitable for large-scale systems of order 10^5 or higher.

Some recent work in relation to Balanced Truncation for large-scale sparse systems has been done by Gugercin and Li [4] and for parallel model reduction of systems up to the size of O(10^4) by Benner et al. [5].

In this paper, we address the combination of two methods – the first is based on a Fourier Series Expansion [6]. The full-size model is simulated (or measurements can be taken from the system if a physical representation is present) and an intermediate model is formed using the Fourier Series Expansion. Guaranteed stability and causality is assured with this model. The second stage of the technique is the application of standard Balanced Truncation [7] to further reduce the model and extract a compact model with a global error bound.

The proposed method is applied to two examples and results will highlight the efficiency and efficacy of the proposed method.

II. Fourier Series Expansion

The Fourier Series Expansion was first introduced in [6] and is summarized here for completeness. Consider a large-scale linear system. The goal is to determine a reduced-order model that can be used in subsequent design or analytical work. Let the system be described by a transfer function $H(\omega)$ obtained from simulation of the full system. Suppose $H(\omega)$ is nonzero for $|\omega| \in [0, \omega_m]$ where ω_m is assumed to be large, but finite. Also, assume $H(\omega) = H^*(-\omega)$. Then $\operatorname{Re} H(\omega)$ may be expanded in a Fourier series as follows, bearing in mind that it must be an even function of frequency.

$$\operatorname{Re} H(\omega) = \sum_{k=0}^{\infty} a_k \cos k\tilde{\omega} \tag{1}$$

where $\tilde{\omega} = \pi\omega / \omega_m$. The expression in (1) describes an even function, defined for $\omega \in [-\omega_m, \omega_m]$ (i.e. $\tilde{\omega} \in [-\pi, \pi]$). Assuming causality or to enforce causality, the expression for $\operatorname{Im} H(\omega)$ may be obtained from (1) via the Kramers-Kronig relations (Hilbert transform) [9], [10]:

$$\operatorname{Im} H(\omega) = -\sum_{k=0}^{\infty} a_k \sin k\tilde{\omega} \tag{2}$$

From (1) and (2), it follows that

$$H(\omega) = \sum_{k=0}^{\infty} a_k e^{-jk\tilde{\omega}} \tag{3}$$

for $\omega \in [-\omega_m, \omega_m]$.

The representation of the output in the time-domain may be obtained by an Inverse Fourier Transform. The output caused by an (arbitrary) input $x(t)$ defined for $t>0$, (i.e. input signal $x(t)\theta(t)$ with Fourier image $X(\omega)$ where $\theta(t)$ is the unit step-function), is:

$$\begin{aligned} y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} Y(\omega) d\omega \\ &= \sum_{k=0}^{\infty} a_k \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j(t-\tilde{k})\omega} X(\omega) d\omega \\ &= \sum_{k=0}^{\infty} a_k x(t-\tilde{k})\theta(t-\tilde{k}) \end{aligned} \tag{4}$$

where $\tilde{k} = \pi k / \omega_m$.

Therefore, once the set of FSE coefficients $\{a_k\}$ is obtained from the frequency-domain simulations (i.e. from (1)), then the response for an arbitrary input may be readily determined from (4).

To determine the set of FSE coefficients, let $H(\omega)$ be obtained at a number of points, ω_i:

$$F_i^{(1)} = \operatorname{Re} H(\omega_i), \quad i = 1,2,\ldots,N_1 \tag{5}$$

$$F_i^{(2)} = \operatorname{Im} H(\omega_i), \quad i = 1,2,\ldots,N_2 \tag{6}$$

where N_1 is the number of real parts of the data points and N_2 is the number imaginary parts of the data points.

Then let a be the set of real coefficients $a = [a_0 \quad a_1 \quad \dots \quad a_N]^T$.

Let $M_{ik}^{(1)} = \cos k\tilde{\omega}_i$, $M_{ik}^{(2)} = -\sin k\tilde{\omega}_i$ and $\tilde{\omega}_i = \pi\omega_i / \omega_m$ where k=1,.., N.

Then from (1) and (2):

$$F^{(1)} = M^{(1)} a + E^{(1)} \tag{7}$$

$$F^{(2)} = M^{(2)} a + E^{(2)} \tag{8}$$

$E^{(1,2)}$ represent the errors that arise due to limiting the summation in (1)-(4) to a finite number of terms, N.

(7) and (8) may be merged to yield:

$$F = Ma + E \tag{10}$$

with

$$F = \begin{bmatrix} F^{(1)} \\ F^{(2)} \end{bmatrix}, \quad M = \begin{bmatrix} M^{(1)} \\ M^{(2)} \end{bmatrix}, \quad E = \begin{bmatrix} E^{(1)} \\ E^{(2)} \end{bmatrix}.$$

and the minimal error, $E^T E$, for (10) is achieved with:

$$a = \left(M^T M\right)^{-1} M^T F \tag{11}$$

III. Formation of a Reduced-Order State Space

Once the vector of coefficients a is found from (11), the next step is to convert this representation into an intermediate discrete-time state-space model. The approach employed follows from Gugercin and Willcox [11, 12]. However, in contrast to this work, they form an intermediate state-space model from the original state-space model using a Krylov technique for Fourier Model Reduction. If the sampling time T of the intermediate discrete-time model is set as $T = \tilde{\omega} = \frac{\pi}{\omega_m}$, then:

$$H_{\text{int}}(z) = C_r (zI - A_r)^{-1} B_r + D_r \tag{12}$$

Where:

$A_r = [e_2, e_3, \cdots, e_N, 0]$, $B_r = [e_1]$, $C_r = [a_1, a_2, \cdots, a_N]$, $D_r = [a_0]$

e_i denotes the ith unit vector of R^N and 0 is a vector of zeros and a_i are the coefficients determined in (11). $z = e^{j\omega T}$. H_{int} is the transfer function of the intermediate reduced system. Because the state-space model in (12) is derived from a Fourier Series representation of the linear system, it is guaranteed to be stable. However, the Fourier Series representation may contain redundant information so this is why at this point, Balanced Truncation may be applied to the representation in (12). Because of the form of (12), the Hankel matrix is known explicitly:

$$\Gamma = \begin{bmatrix} a_1 & a_2 & \cdots & \cdots & a_N \\ a_2 & a_3 & \cdots & a_N & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_N & 0 & \cdots & \cdots & 0 \end{bmatrix} \tag{13}$$

The first k singular vectors of Γ, corresponding to the k largest singular values σ_i of Γ, are used to determine a projection matrix V_k. The discrete-time reduced system is then formed as:

$$\begin{aligned} \hat{x}(t+1) &= \hat{A}\hat{x}(t) + \hat{B}u(t) \\ \hat{y}(t) &= \hat{C}\hat{x}(t) + \hat{D}u(t) \end{aligned} \tag{14}$$

where

$\hat{A} = V_k^T A_r V_k$ $\hat{B} = V_k^T B_r$ $\hat{C} = C_r V_k$ $\hat{D} = D_r$

The superscript T denotes the transpose of a matrix. A continuous time reduced-order model can be formed using the inverse bilinear transform.

Note that use of Balanced Truncation yields the following expression for the error bound

$$\left\| H_{\text{int}} - \hat{H} \right\|_\infty \le 2 \sum_{i=k+1}^{N} \sigma_i \tag{15}$$

$\hat{H}$ is the transfer function of the final reduced model.

IV. EXAMPLES

The first example considered is that of the two symmetric coupled dispersive strip-lines as described in [8]. The reduction procedure in the previous sections is applied to measurements or simulations from the original complex system.

The red solid line in Fig. 1 shows the measured real part of the coefficient scattering S_{12} of the lossy coupled lines.

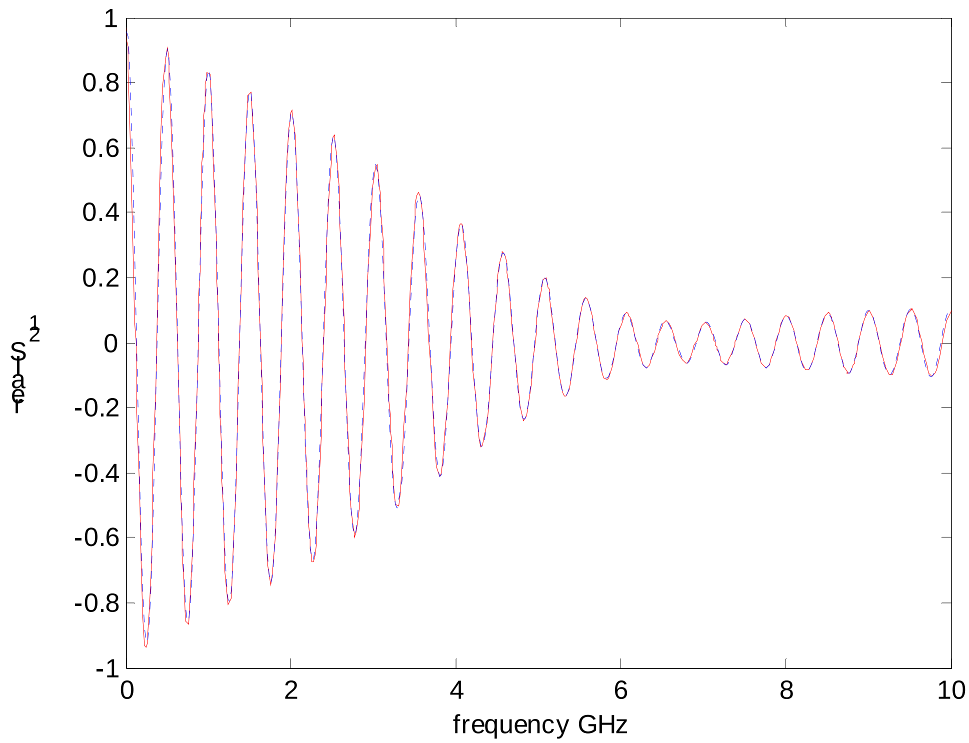

Fig. 1 *Real (S_{12}) –measured data (red solid lines), reduced model (blue dashed lines)*

The blue dashed line shows the corresponding result obtained with the proposed reduction method. The size of the reduced model is determined by the number of singular vectors selected to form the projection matrix V_k. In our work, we select vectors corresponding to the singular values σ_k such that $\sigma_k / \sigma_{\max} > value$. $\sigma_{\max}$ is the largest singular value and *value* is set by the user so as to achieve a certain level of accuracy. For our simulations for this example, *value=0.1*. As can be seen from the result in Fig. 1, a high degree of accuracy is obtained with a reduced-order state-space model of size k=29.

The second example taken is the sample interconnect network in Fig. 1 of [6].

The size of the reduced model is determined by selecting singular vectors to form the projection matrix V_k for which the corresponding singular values σ_k are such that $\sigma_k / \sigma_{\max} > 0.04$.

Fig. 2 compares the transient output from a full model and that from the reduced-order model of size k=10 obtained with the method detailed above. As is evident, the result from the reduced model captures all of the essential behaviour.

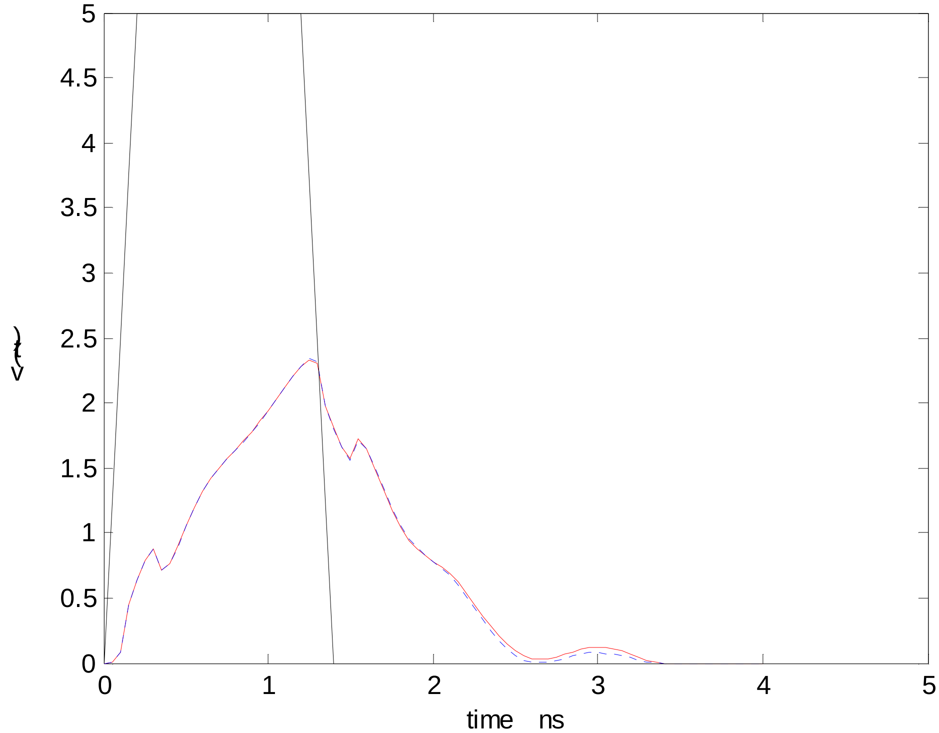

Fig. 2 *Transient results (input=black line) (red line='measured' output) (blue dashed line=output from reduced model)*

V. CONCLUSIONS

The paper has proposed a two-stage method for forming a reduced-order model of large-scale systems. The method combines two techniques, a Fourier Series approach and Balanced Truncation. The method achieves a high degree of accuracy and eliminates any redundant information from the reduced model and thus improves on computational efficiency.

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